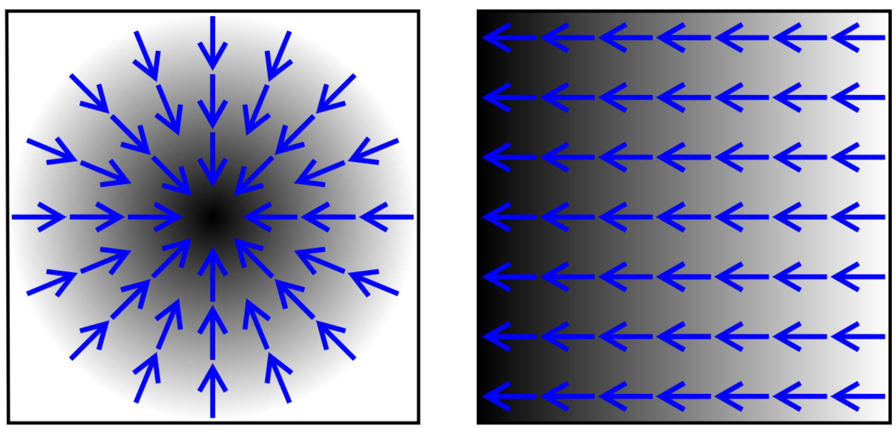
Review

Typical Notation

$$``\Delta = \underbrace{\nabla \cdot}_{\text{divergence}} \underbrace{\nabla}_{\text{gradient}}"$$

More later...



Gradient operator:

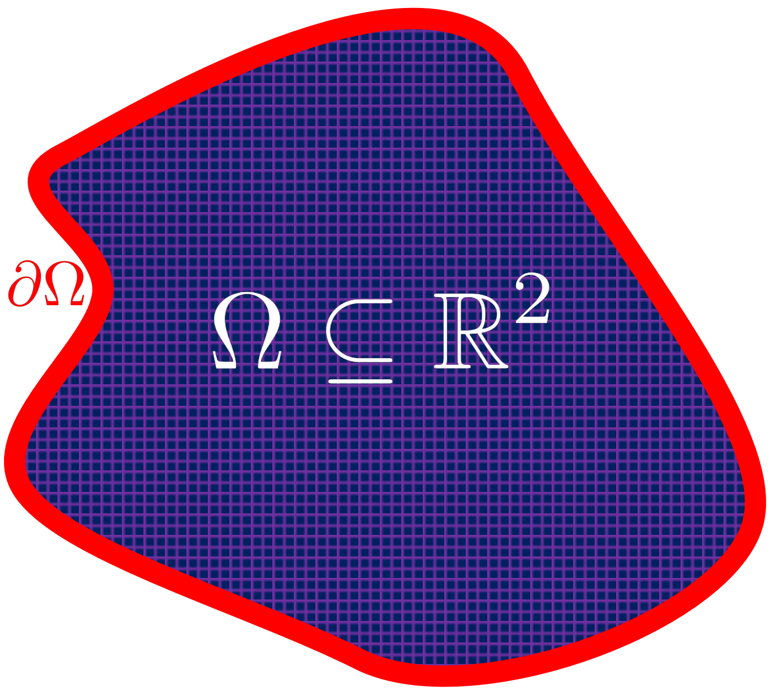
$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

http://www.gamasutra.com/db_area/images/feature/4164/figy.png, https://en.wikipedia.org/wiki/Gradient

Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

"Dirichlet boundary conditions"



$$\mathcal{L}[f] := -\Delta f$$

$$\langle f, g \rangle := \int_\Omega f(x)g(x)\,dx$$

On board:

1. Positive: $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. Self-adjoint: $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

Proof

Proof of 1

$$\langle f, \mathscr{L}[f] \rangle = \int_{\Omega} f(-\nabla \cdot \nabla f)\, dV = \int_{\partial\Omega} f(-\nabla f \cdot \overrightarrow{n})\, dS + \int_{\Omega} \nabla f \cdot \nabla f\, dV = \int_{\Omega} \nabla f \cdot \nabla f\, dV \geq 0$$

where the second equality follows from Green formula, and the third equality follows from $f|_{\partial\Omega} \equiv 0$

Proof of 2

$$\langle f, \mathscr{L}[g] \rangle = \int_{\Omega} f(-\nabla \cdot \nabla g)\, dV = \int_{\partial\Omega} f(-\nabla g \cdot \overrightarrow{n})\, dS + \int_{\Omega} \nabla f \cdot \nabla g\, dV = \int_{\Omega} \nabla f \cdot \nabla g\, dV$$

where the second equality follows from Green formula, and the third equality follows from $f|_{\partial\Omega} \equiv 0$

Similarly, $\langle \mathscr{L}[f], g \rangle = \int_{\Omega} \nabla g \cdot \nabla f\, dV$

It also shows $\langle f, \mathscr{L}[g] \rangle = \int_{\Omega} \nabla f \cdot \nabla g\, dV$

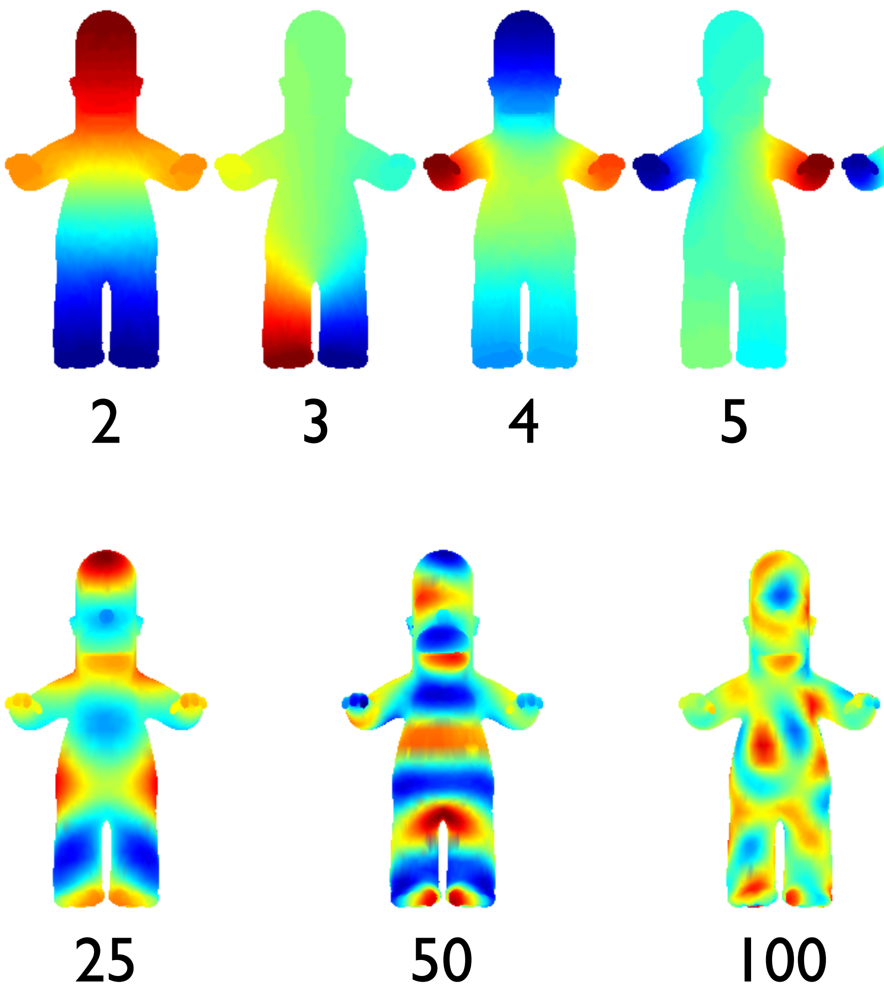
Laplacian(-Bertrami) Operator Diagonalizable!

Theorem. Let $H \neq 0$ be an infinite-dimensional, separable Hilbert space and let $K \in L(H)$ be compact and self-adjoint. Then, there exists a countable orthonormal basis of H consisting of eigenvectors of K.

Hilbert space: Space with inner product
Separable: Admits countable, dense subset
Compact operator: Bounded sets to relatively compact sets
Self-adjoint: $\langle Kv, w\rangle = \langle v, Kw\rangle$

https://www.math.ku.edu/~matjohn/Teaching/S16/Math951/HilbertSchmidt.pdf

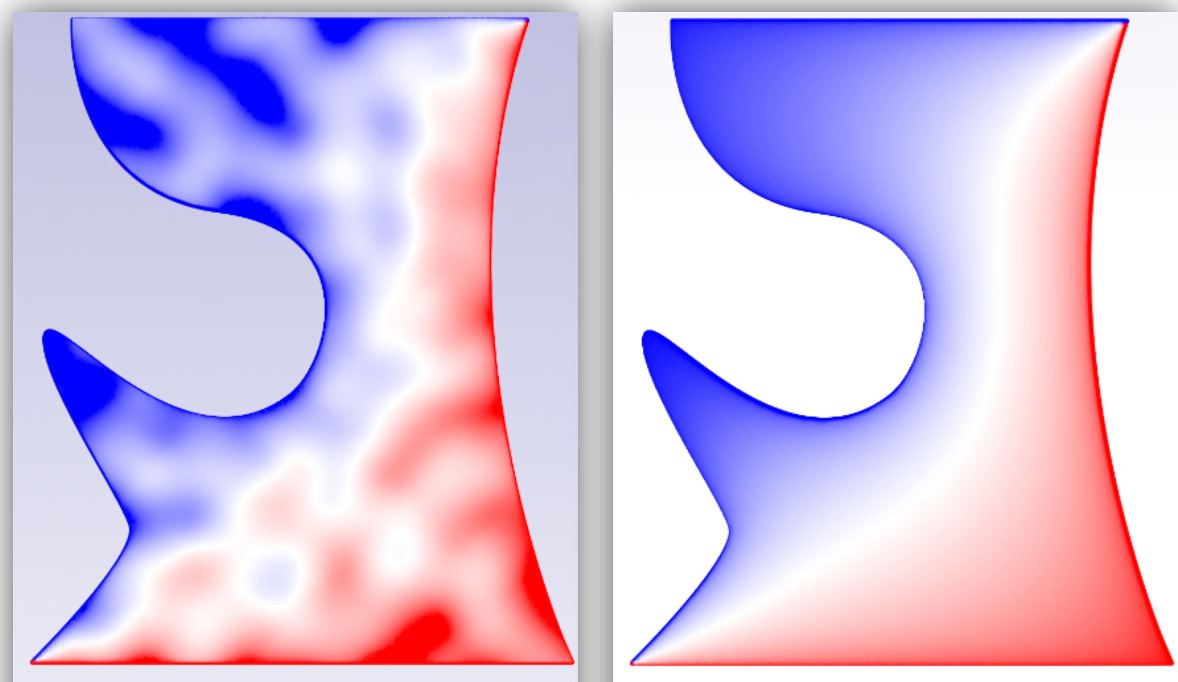
Eigenhomers



What is smallest eigenvalue?

Dirichlet Energy

$$E[f] := \int_\Omega \langle \nabla f, \nabla f \rangle \, dA$$



non-smooth $f(x)$ solution $\Delta f = 0$

Images made by E.Vouga

On board:

$$\min_f E[f]$$

$$\text{s.t. } f|_{\partial\Omega} = g$$

$$\Delta f \equiv 0$$

"Laplace equation"
"Harmonic function"

Proof

We use variational method to derive.

Lagrangian: $\mathbb{L}[f] = \frac{1}{2}\int \langle \nabla f, \nabla f \rangle + \int_{\partial\Omega} \lambda(x)(f(x) - g(x))$

So

$$\delta\mathbb{L}[f] = \mathbb{L}[f+\delta h] - \mathbb{L}[f] = \int_{\Omega} \langle \nabla f, \nabla \delta h \rangle + \int_{\partial\Omega} \lambda(x)\delta h(x) = \{\int_{\partial\Omega} \delta h(\nabla f \cdot \overrightarrow{n}) - \int_{\Omega} \delta h(\nabla \cdot \nabla f)\} + \int_{\partial\Omega} \lambda(x)\delta h(x)$$

In the interior of Ω, $\Delta f \equiv 0$ so that $\delta\mathbb{L}[f] = 0$ for any δh

Note: in the derivation we ignored the second-order infinitesimal term $O(\|\delta h\|^2)$

UC San Diego

CSE291-C00

Discrete Laplacian & Its Applications

Instructor: Hao Su

Discrete Laplacian

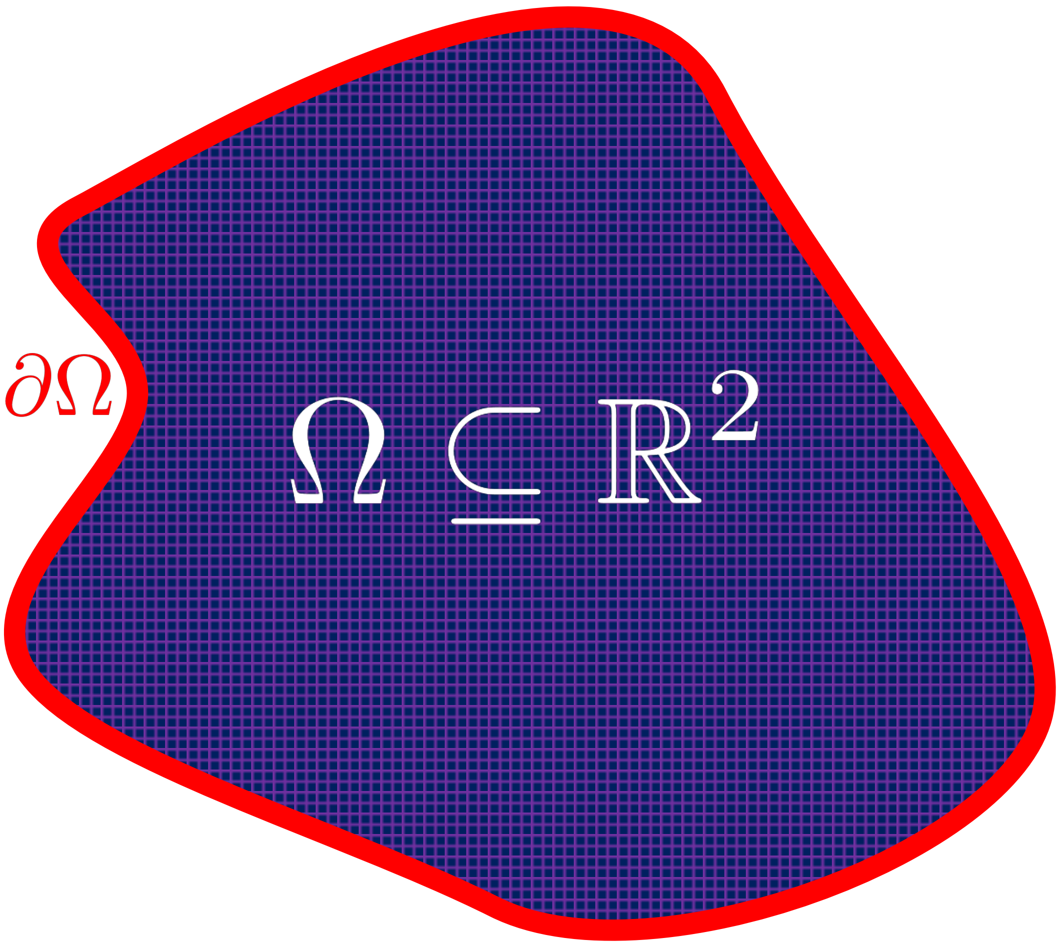
Our Focus

$$f \in C^\infty(M) \longrightarrow \Delta \longrightarrow \Delta f \in C^\infty(M)$$

Computational version?

The Laplacian

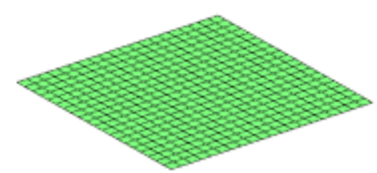
Recall: Planar Region



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



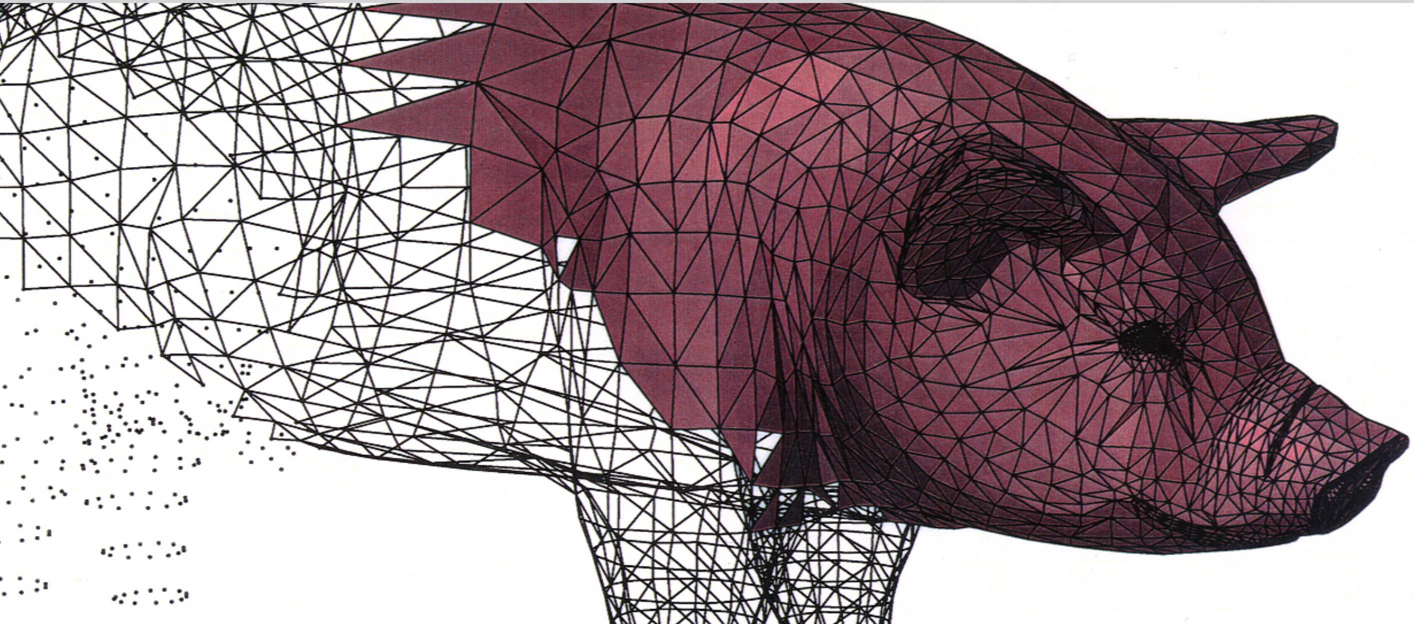
Discretizing the Laplacian

$$\Delta f = \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|}g^{ij}\partial_j f\right)$$

?!

Problem

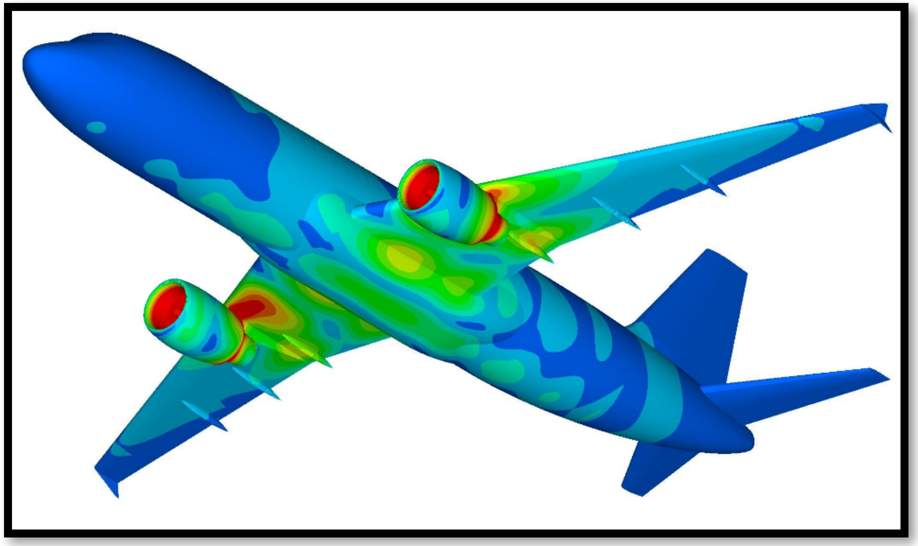
Laplacian is a *differential* operator!



A Principled Approach to Connect Continuous & Discrete Objects

First-order Galerkin

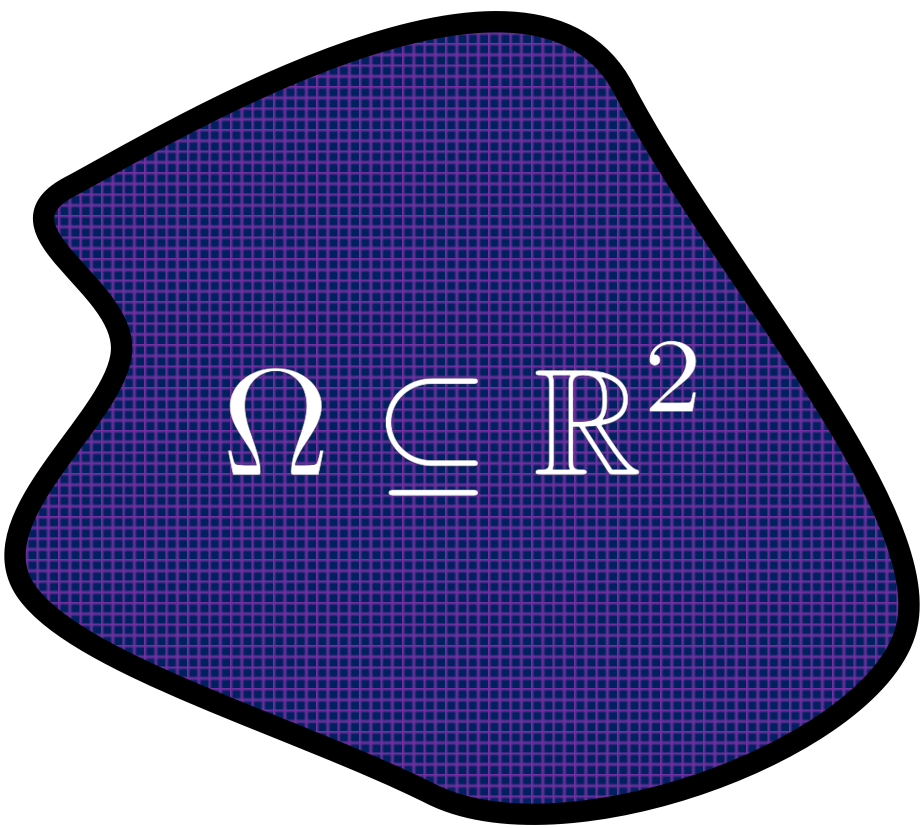
Finite element method (FEM)

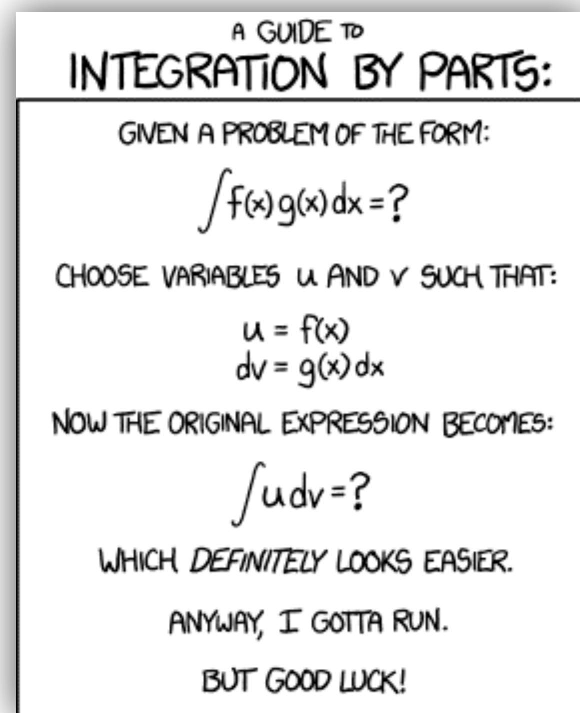


http://www.stressebook.com/wp-content/uploads/2014/08/Airbus_A320_k.jpg

Integration by Parts to the Rescue

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} - \int_\Omega \nabla f \cdot \nabla g \, dA$$





https://xkcd.com/1201/

Slightly Easier?

$$\int_\Omega f\,\Delta g\,dA = \text{boundary terms} - \int_\Omega \nabla f \cdot \nabla g\,dA$$

Laplacian
(second derivative)

Gradient
(first derivative)

Slightly Easier?

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} - \int_\Omega \nabla f \cdot \nabla g \, dA$$

One derivative,
one integral

Gradient
(first derivative)

Kinda-sorta cancels out?

Overview: Galerkin FEM Approach

$$g = \Delta f$$

$$\implies \int \psi g \, dA = \int \psi \Delta f \, dA = -\int (\nabla \psi \cdot \nabla f) \, dA$$

L2 Dual of a Function

Function $f : M \to \mathbb{R}$

$$\downarrow$$

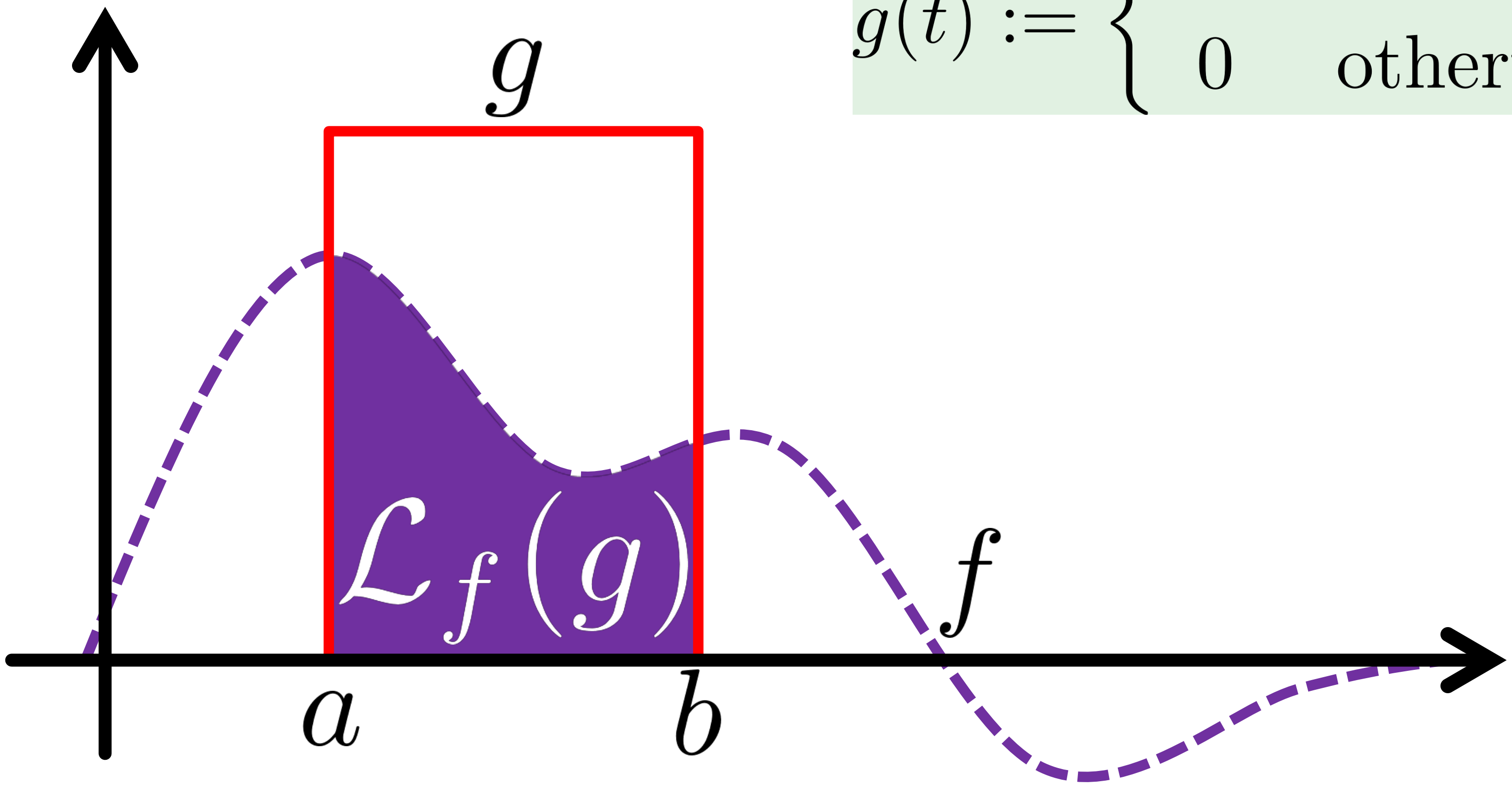
Operator $\mathcal{L}_f : L^2(M) \to \mathbb{R}$

$$\mathcal{L}_f[g] := \int_M f(x)g(x)\, dA$$

"Test function"

Observation

$$g(t) := \begin{cases} 1 & \text{if } t \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



Can recover function from dual

Dual of Laplacian

Space of test functions:

$$\{g \in L^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta f}[g] &= \int_M g \Delta f \, dA \\ &= -\int_M \nabla g \cdot \nabla f \, dA\end{aligned}$$

Use Laplacian without evaluating it!

Galerkin's Approach

Choose one of each:

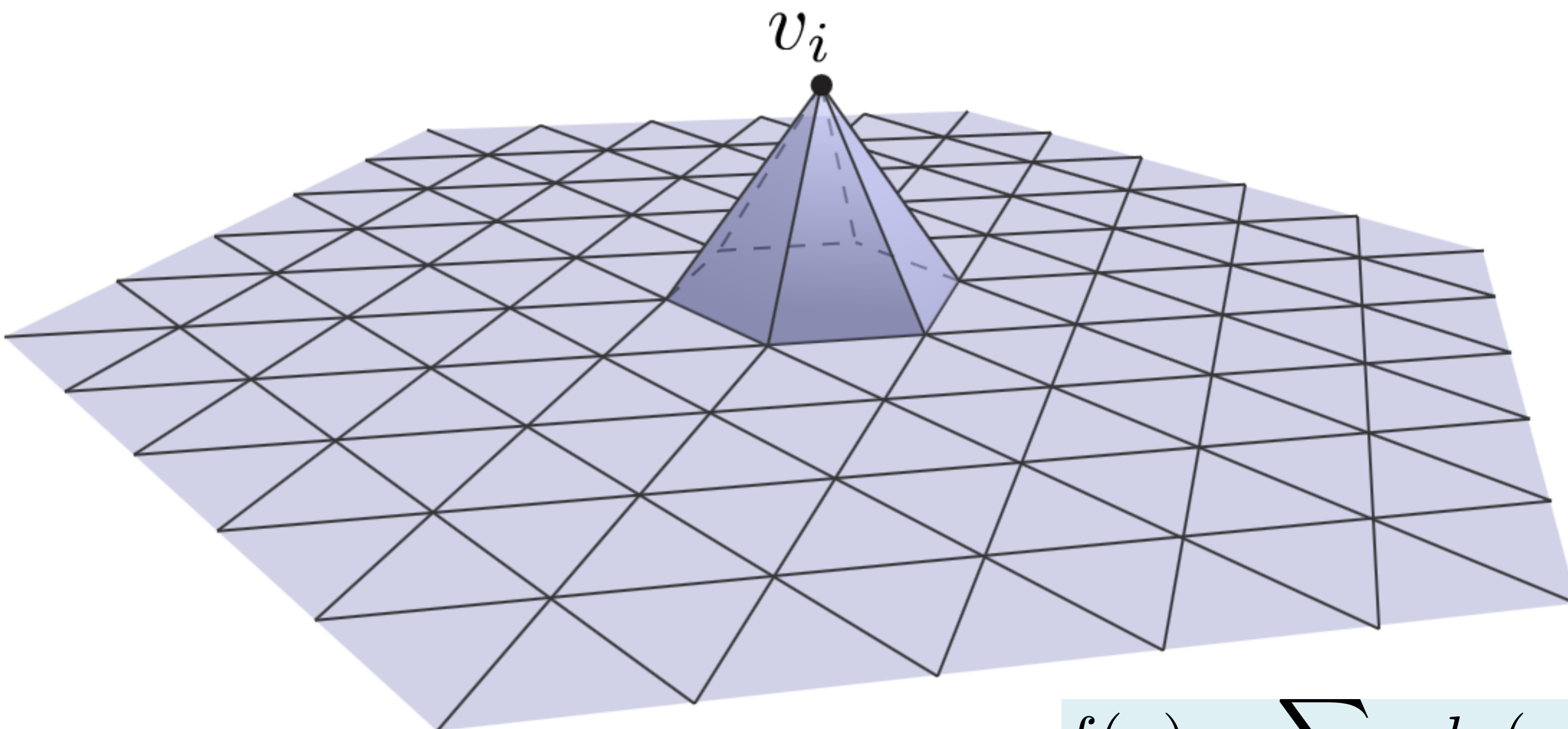
- Function space
- Test functions

Often the same!

One Derivative is Enough

$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \, dA$$

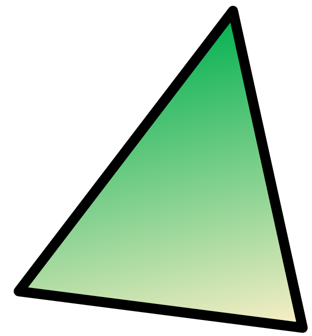
Representing Functions



$$f(x) = \sum_i a_i h_i(x)$$

$$a \in \mathbb{R}^{|V|}$$

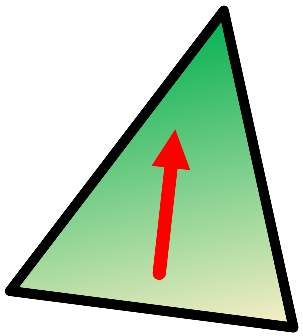
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \, dA$$

Linear combination of hats
(piecewise linear)

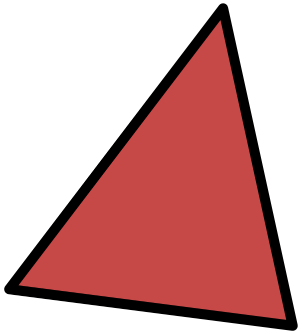
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \, dA$$

One vector per face

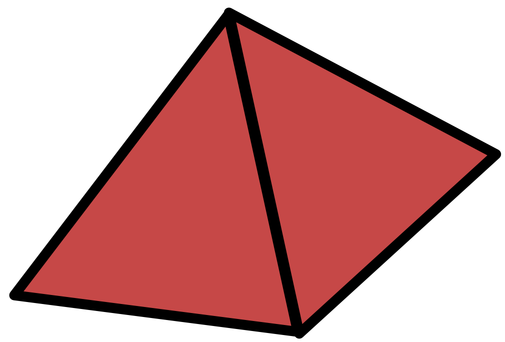
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \, dA$$

One scalar per face

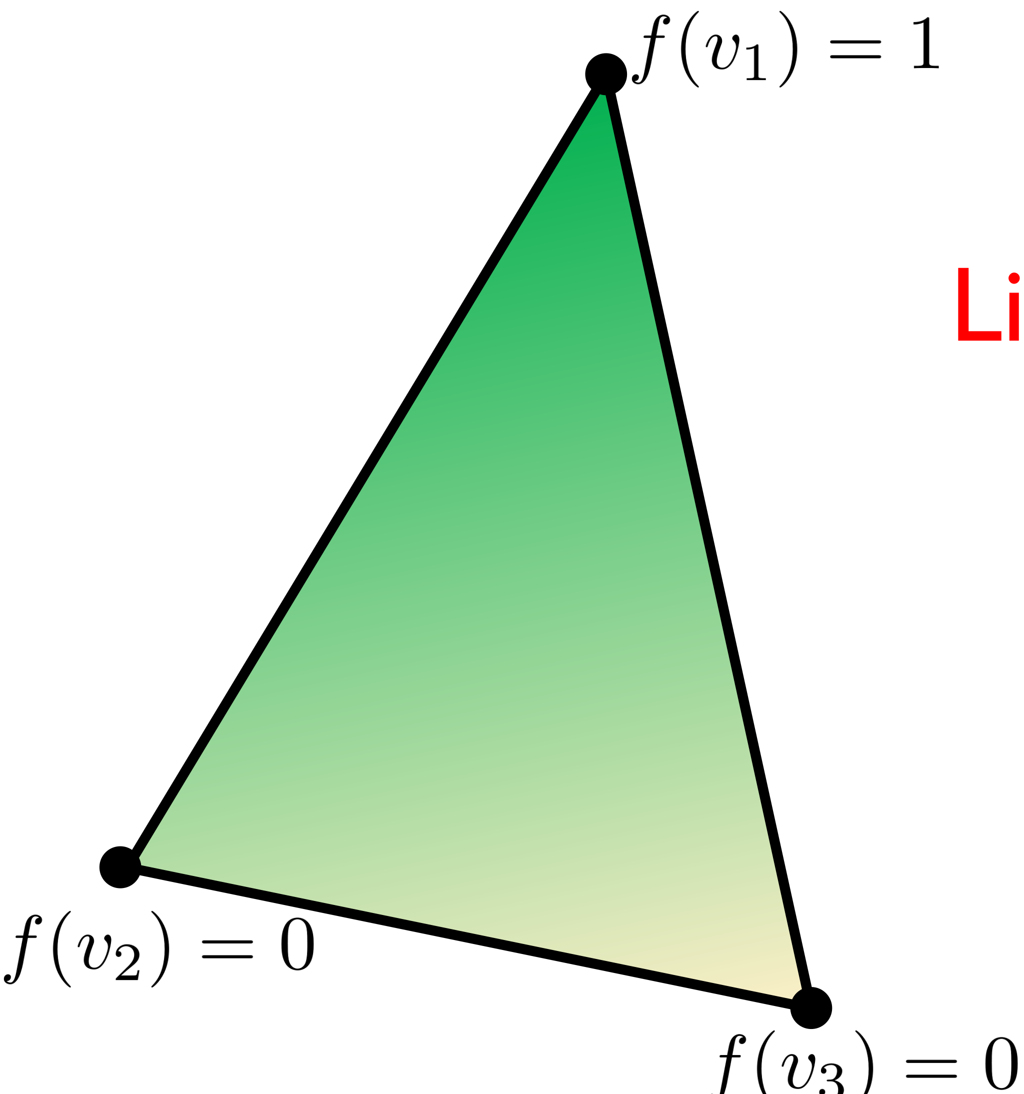
What Do We Need



$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \, dA$$

Sum scalars per face
multiplied by face areas

Gradient of a Hat Function



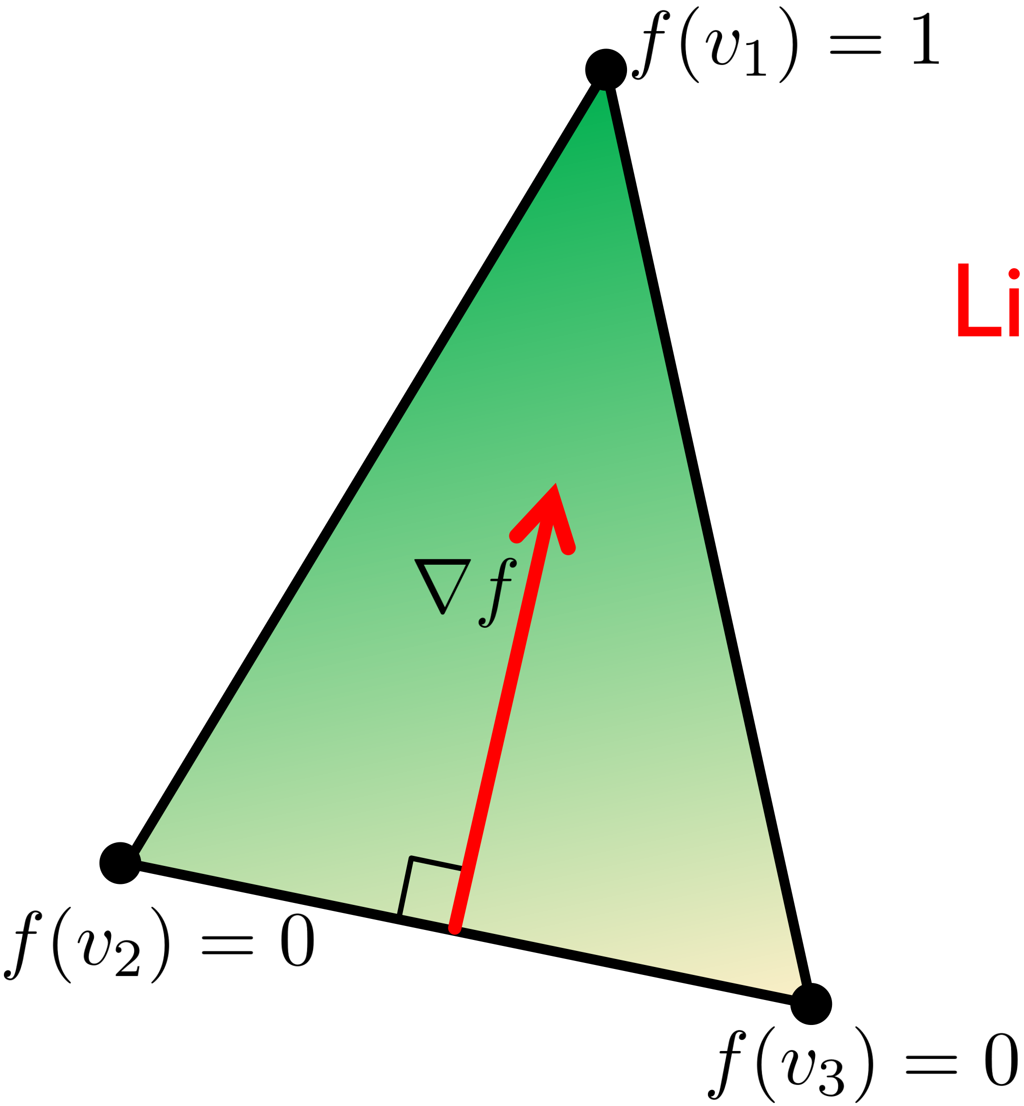
Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

Gradient of a Hat Function



Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

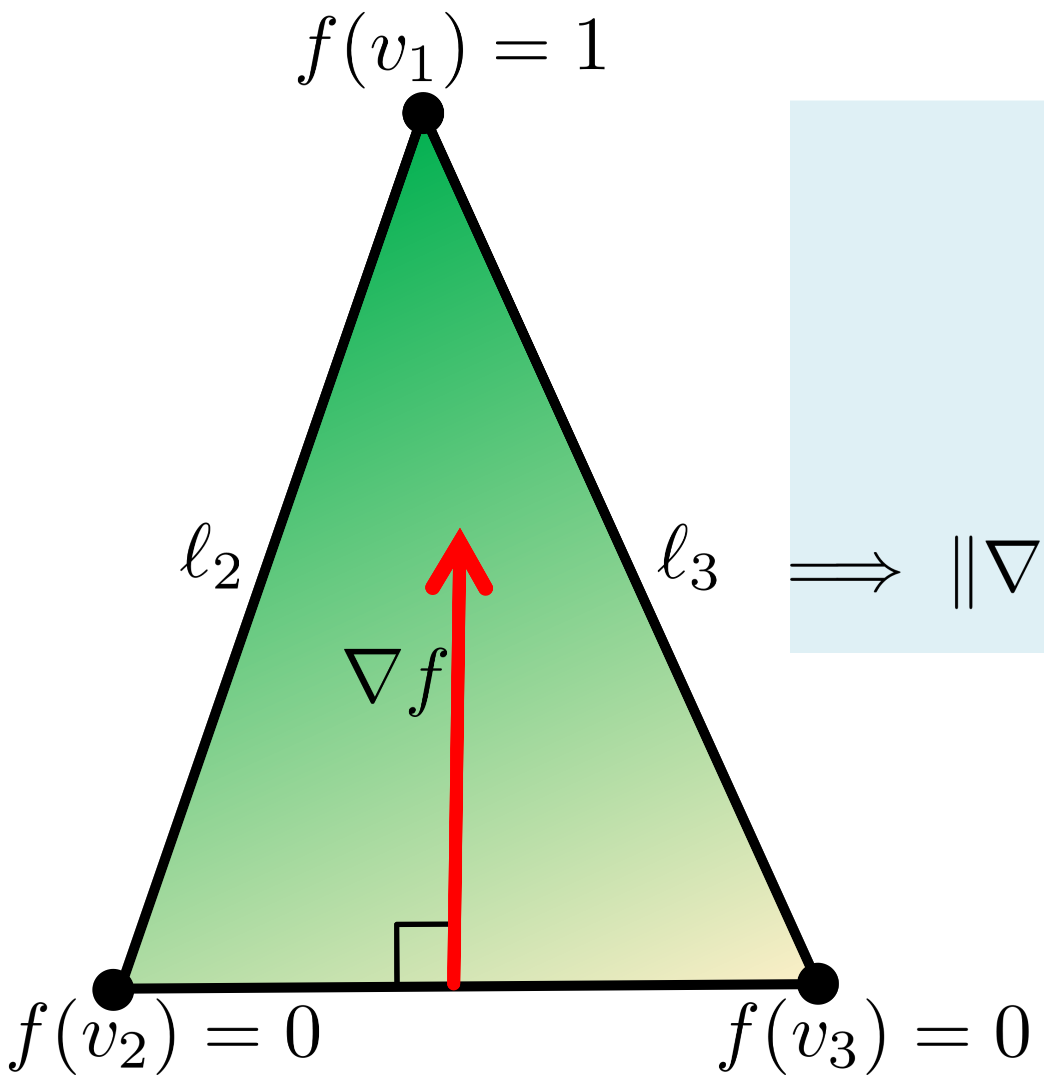
$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

$$\Downarrow$$

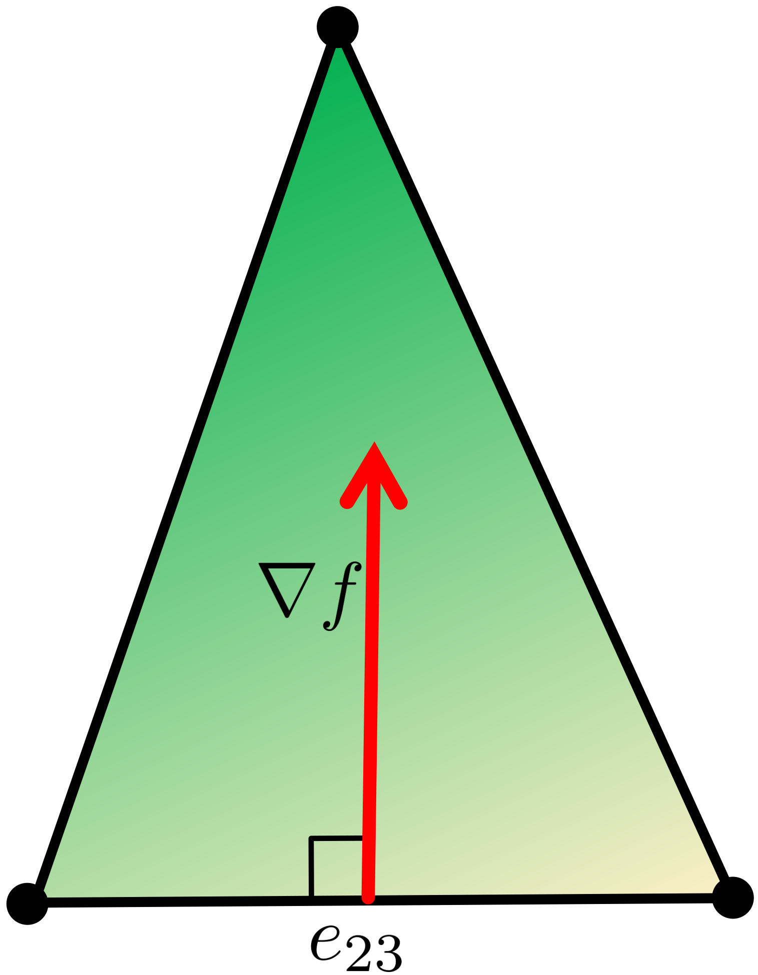
$$\nabla f \cdot (v_2 - v_3) = 0$$

Gradient of a Hat Function



$$
\begin{aligned}
1 &= \nabla f \cdot (v_1 - v_3) \\
&= \|\nabla f\| \ell_3 \cos\left(\frac{\pi}{2} - \theta_3\right) \\
&= \|\nabla f\| \ell_3 \sin\theta_3 \\
\Longrightarrow \quad \|\nabla f\| &= \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}
\end{aligned}
$$

Gradient of a Hat Function



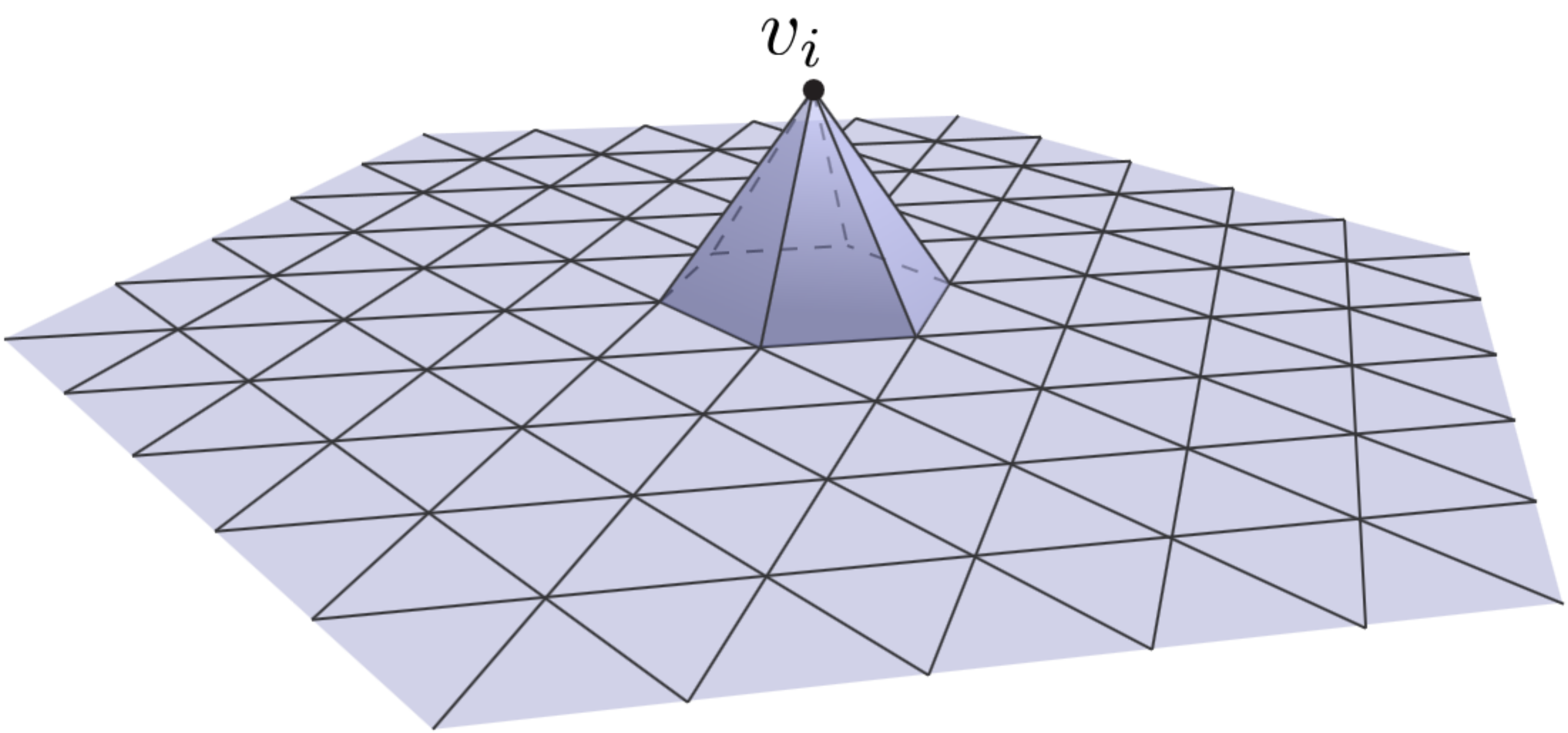
$$\|\nabla f\| = \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}$$

$$\boxed{\nabla f = \frac{e_{23}^{\perp}}{2A}}$$

Length of e_{23} cancels "base" in A

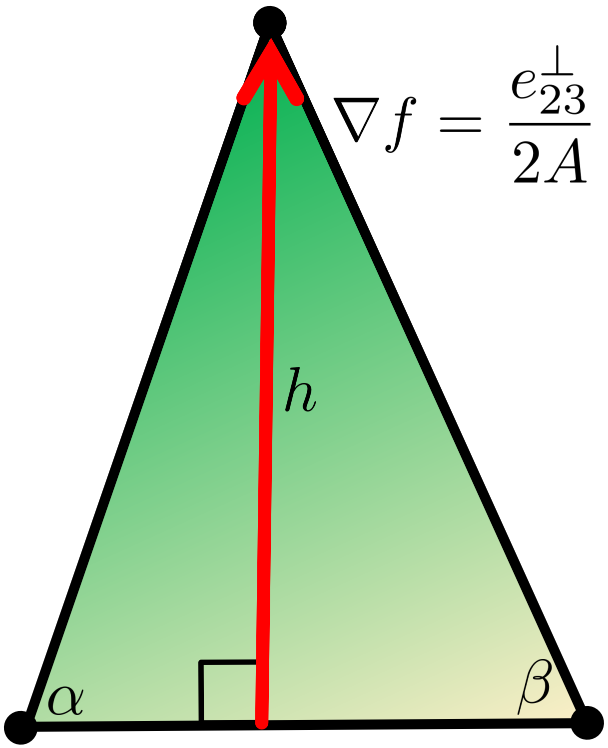
What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \boxed{\nabla g \cdot \nabla f} \, dA$$



What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \boxed{\nabla g \cdot \nabla f}\, dA$$



Case 1: Same vertex

$$\int_T \langle \nabla f, \nabla f \rangle \, dA = A \|\nabla f\| 2$$

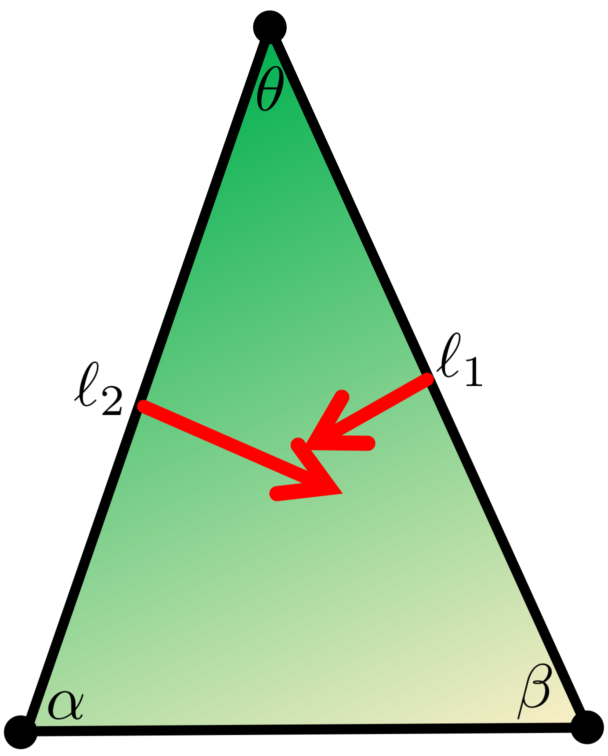
$$= \frac{A}{h^2} = \frac{b}{2h}$$

$$= \frac{1}{2}(\cot \alpha + \cot \beta)$$

What We Actually Need

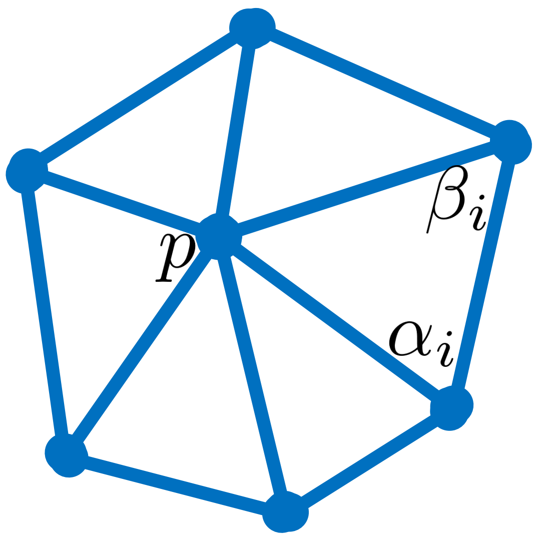
$$\mathcal{L}_{\Delta f}[g] = -\int_M \boxed{\nabla g \cdot \nabla f}\, dA$$

Case 2: Different vertices

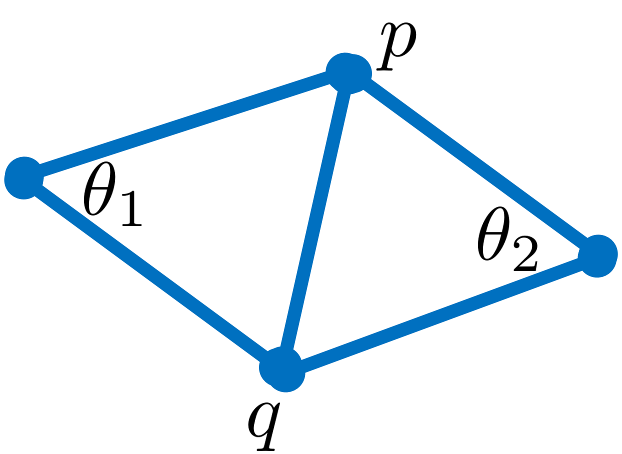


$$\begin{aligned}
\int_T \langle \nabla f_\alpha, \nabla f_\beta \rangle \, dA &= A\langle \nabla f_\alpha, \nabla f_\beta \rangle \\
&= \frac{1}{4A}\langle e_{31}^\perp, e_{12}^\perp \rangle = -\frac{\ell_1 \ell_2 \cos\theta}{4A} \\
&= \frac{-h^2\cos\theta}{4A\sin\alpha\sin\beta} = \frac{-h\cos\theta}{2b\sin\alpha\sin\beta} \\
&= -\frac{\cos\theta}{2\sin(\alpha+\beta)} = -\frac{1}{2}\cot\theta
\end{aligned}$$

Summing Around a Vertex



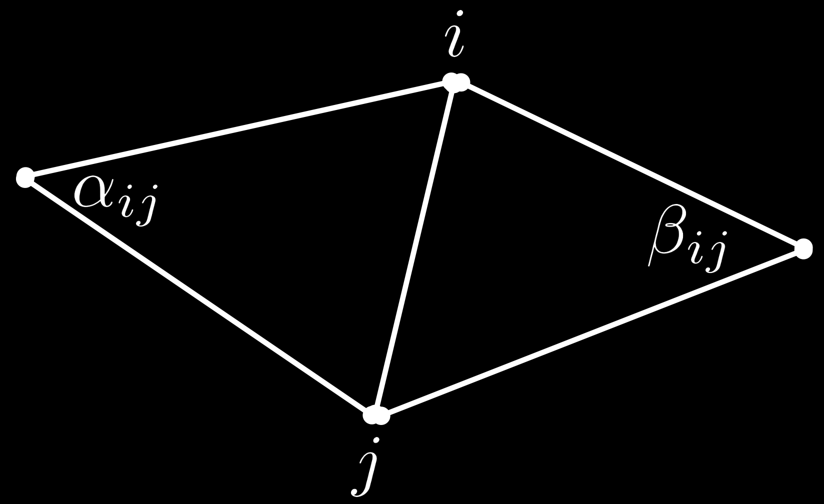
$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$



$$\langle \nabla h_p, \nabla h_q \rangle = \frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

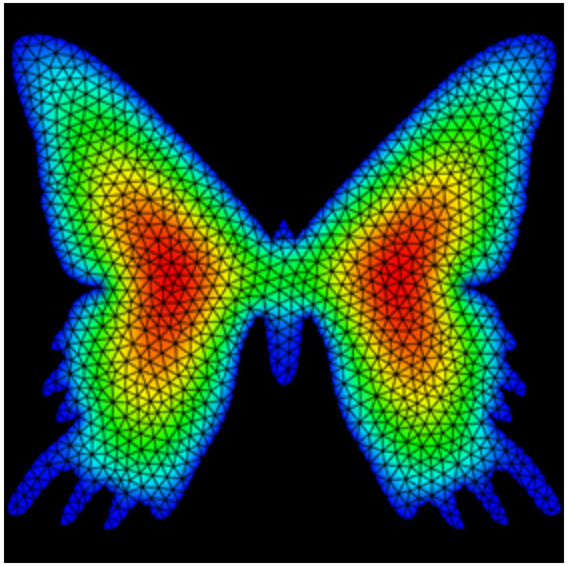
The Cotangent Laplacian

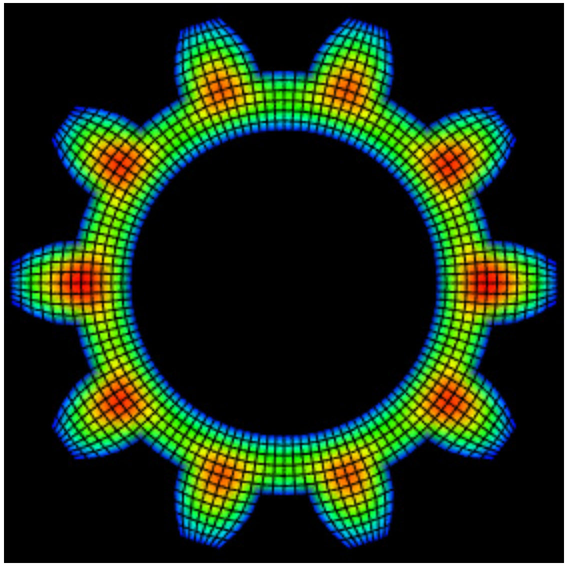
$$L_{ij} = \begin{cases} \frac{1}{2}\sum_{i\sim k}(\cot\alpha_{ik} + \cot\beta_{ik}) & \text{if } i = j \\ -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$



Poisson Equation

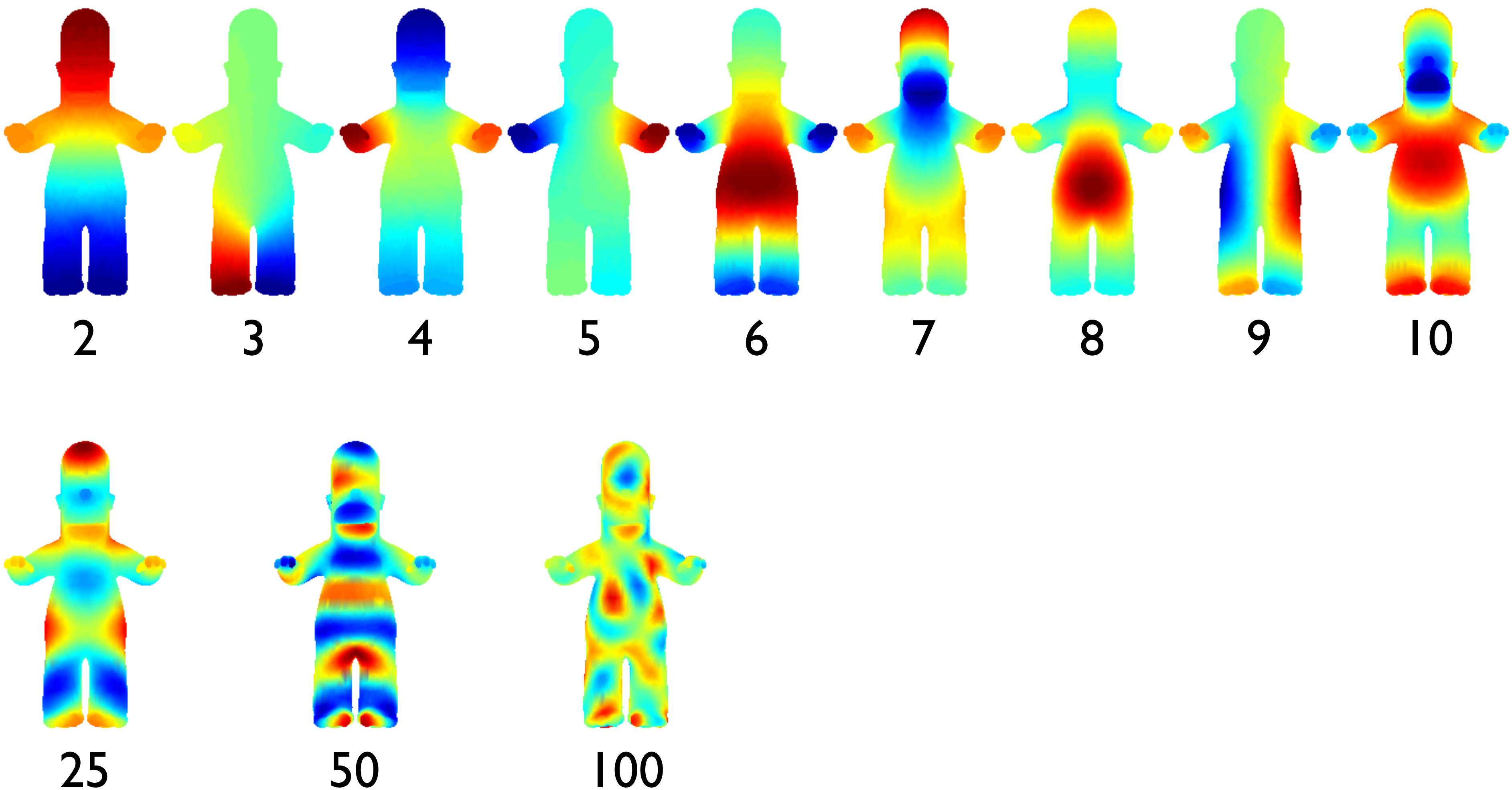
$$\Delta f = g$$





http://nylander.wordpress.com/2006/05/24/finite-element-method-fem-solution-to-poisson%E2%80%99s-equation-on-triangular-mesh/

Eigenhomers



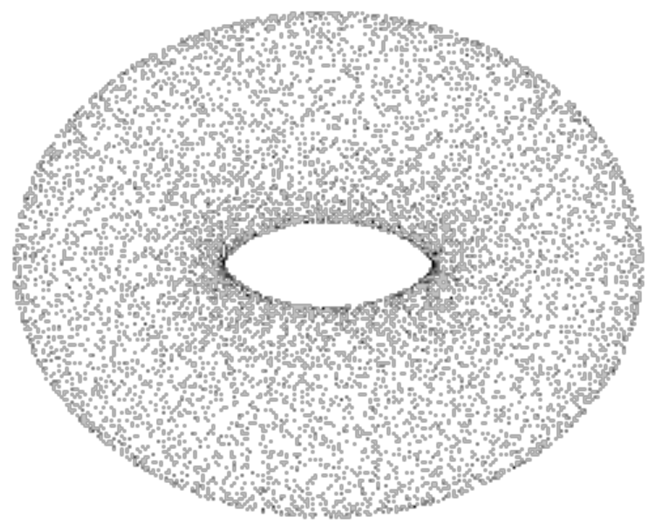
Point Cloud Laplace: Easiest Option

$$W_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{t}\right)$$

$$D_{ii} = \sum_j W_{ji}$$

$$L = D - W$$

$$Lf = \lambda D f$$



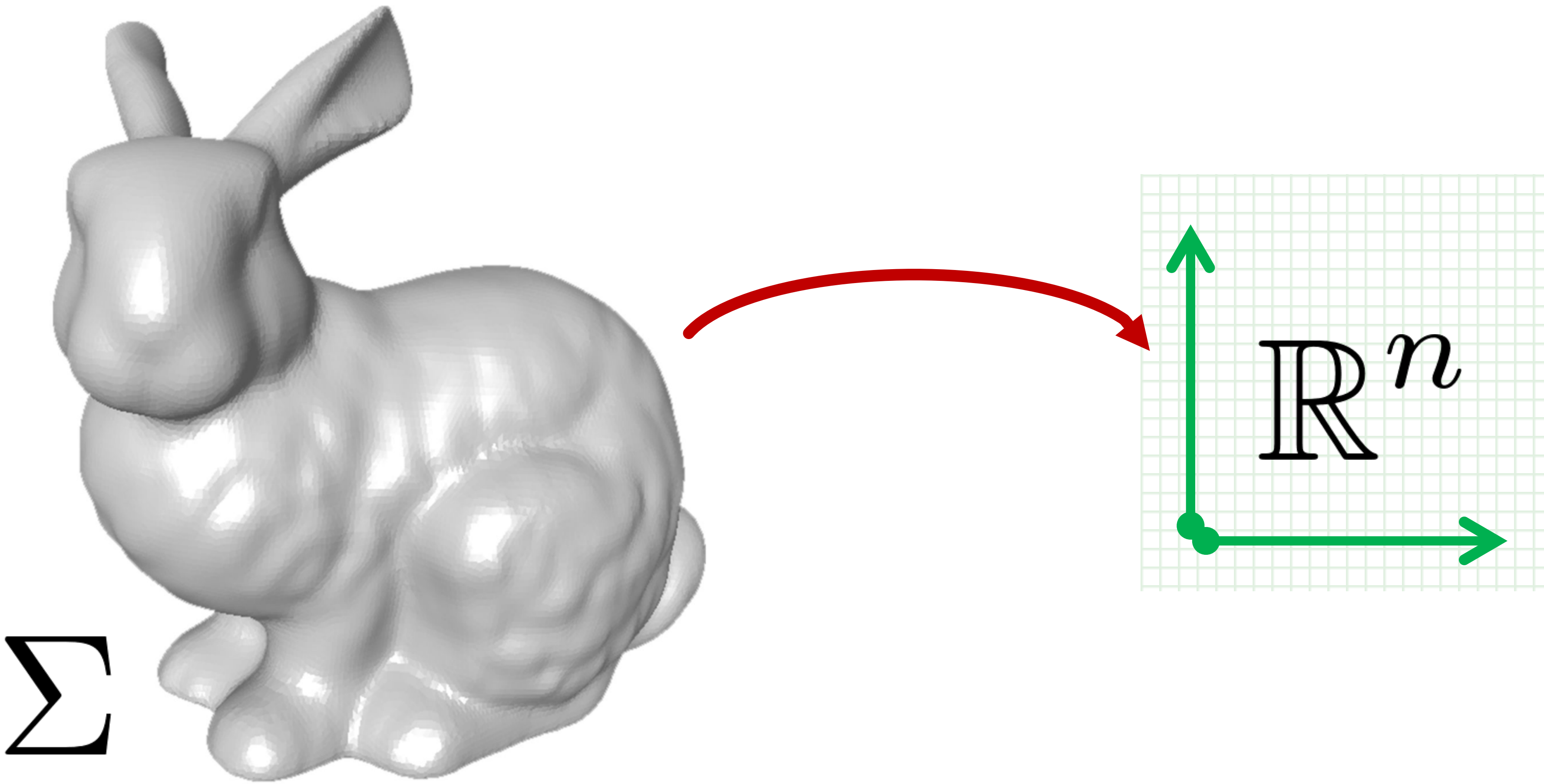
"Laplacian Eigenmaps for Dimensionality Reduction and Data Representation"
Belkin & Niyogi 2003

Applications of Laplacian: Intrinsic Shape Descriptor

Why Study the Laplacian?

- Encodes intrinsic geometry
  Edge lengths on triangle mesh, Riemannian metric on manifold
- Multi-scale
  Filter based on frequency
- Geometry through linear algebra
  Linear/eigenvalue problems, sparse positive definite matrices
- Connection to physics
  Heat equation, wave equation, vibration, …

Example Task: Shape Descriptors



http://liris.cnrs.fr/meshbenchmark/images/fig_attacks.jpg

Pointwise quantity

Isometry Invariance: Hope

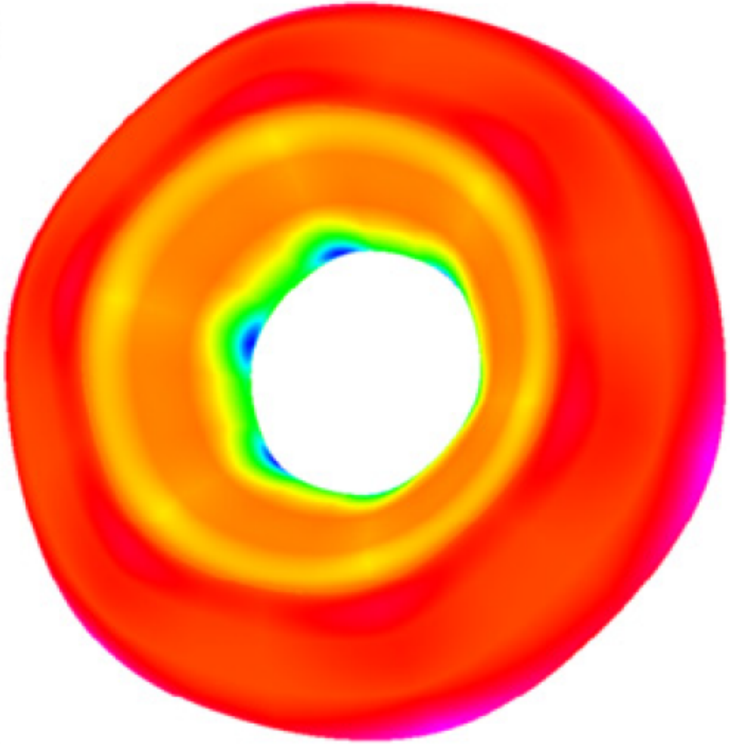
http://www.flickr.com/photos/melvinvoskuijl/galleries/72157624236168459

Descriptor Tasks

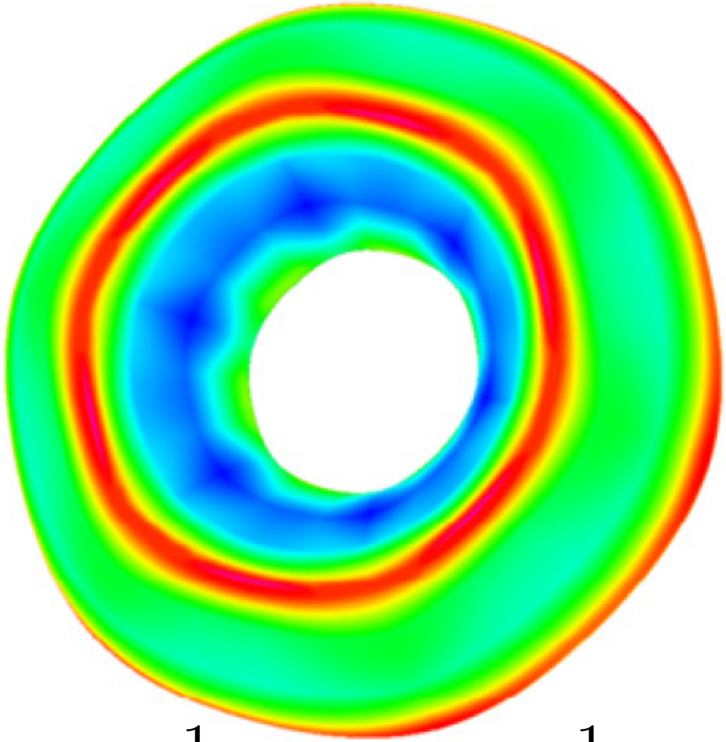
- Characterize local geometry
  Feature/anomaly detection

- Describe point's role on surface
  Symmetry detection, correspondence

Descriptors We've Seen Before



$$K := \kappa_1 \kappa_2 = \det \mathrm{I\!I}$$



$$H := \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} \operatorname{tr} \mathrm{I\!I}$$

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian and mean curvature

Desirable Properties

- **Distinguishing**

  Provides useful information about a point

- **Stable**

  Numerically and geometrically

- **Intrinsic**

  No dependence on embedding

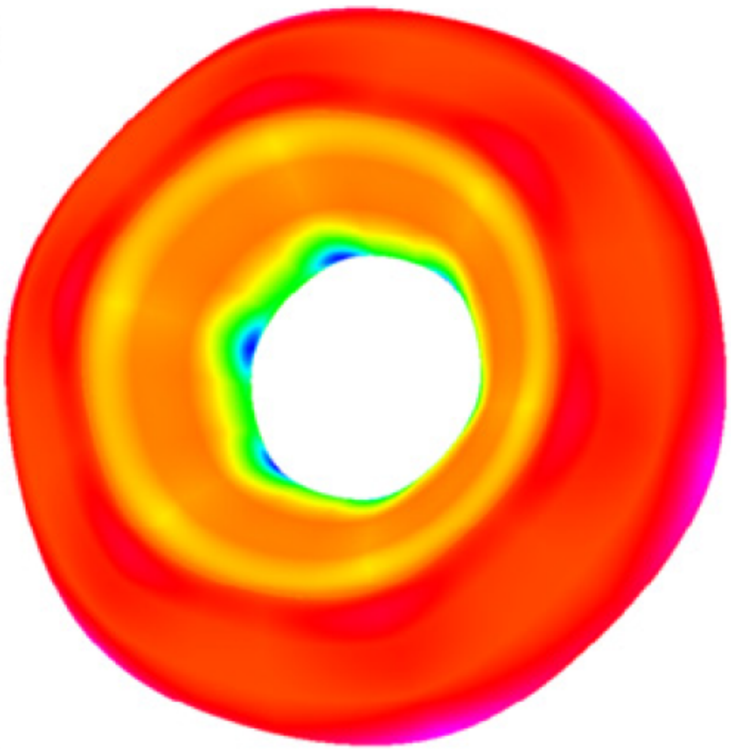
Sometimes undesirable!

Intrinsic Descriptors

Invariant under

- Rigid motion
- Bending without stretching

Intrinsic Descriptor



$$K := \kappa_1 \kappa_2 = \det \mathrm{I\!I}$$

Theorema Egregium

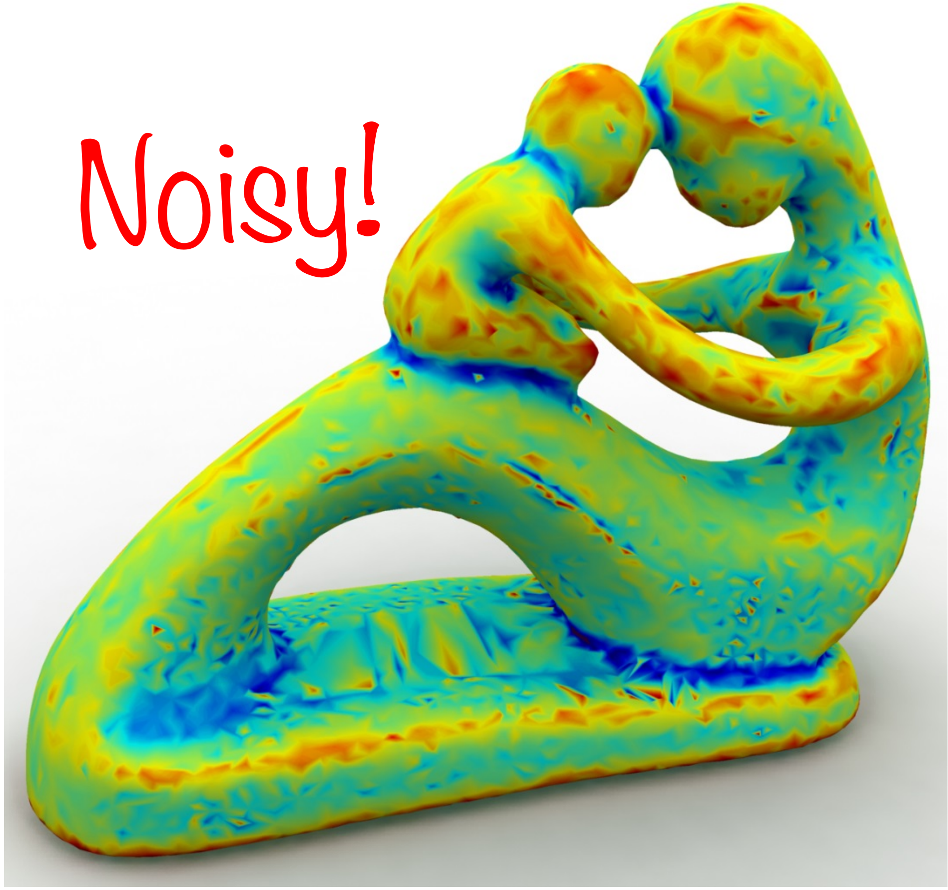
("Totally Awesome Theorem"):

Gaussian curvature is intrinsic.

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian curvature

End of the Story?



$$K = \kappa_1 \kappa_2$$

Second derivative quantity

End of the Story?

Looks the same!

Why not just add mean curvature?

http://www.integrityware.com/images/MerceedesGaussianCurvature.jpg

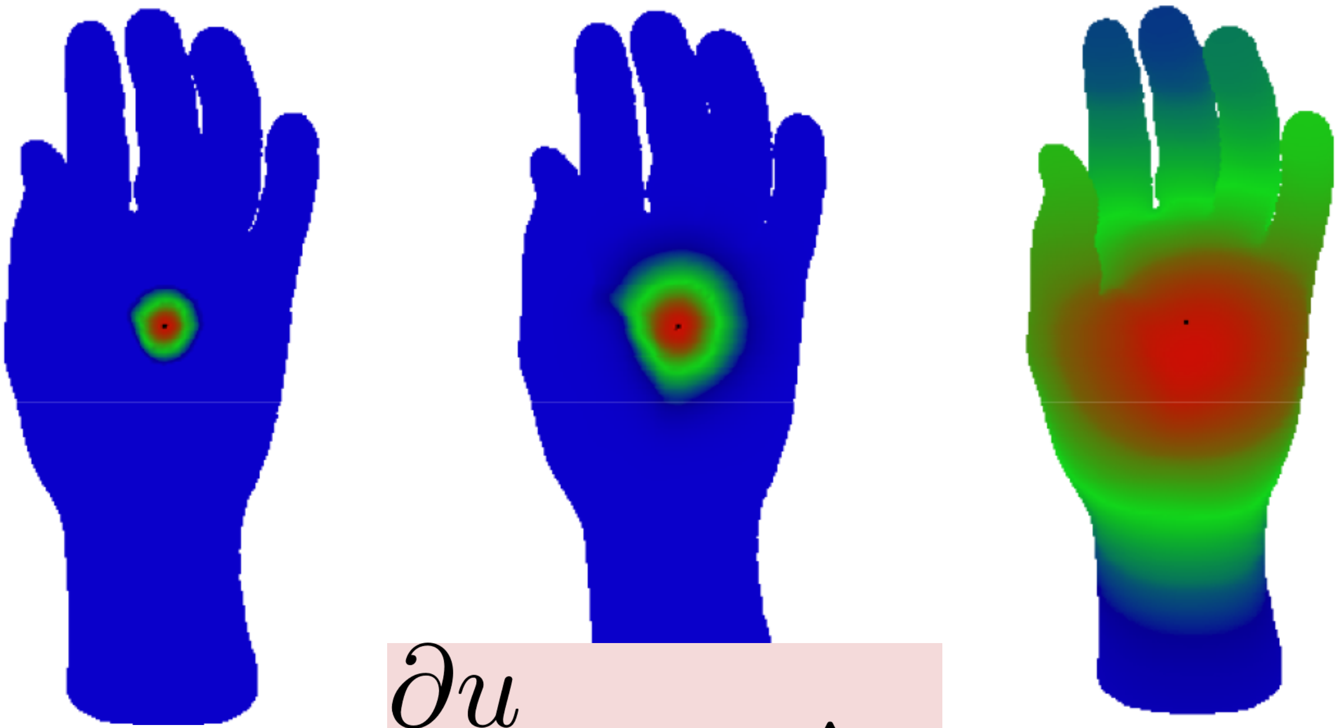
Non-unique

Desirable Properties

Incorporates neighborhood information in an intrinsic fashion

Stable under small deformation

Recall: Connection to Physics



$$\frac{\partial u}{\partial t} = -\Delta u$$

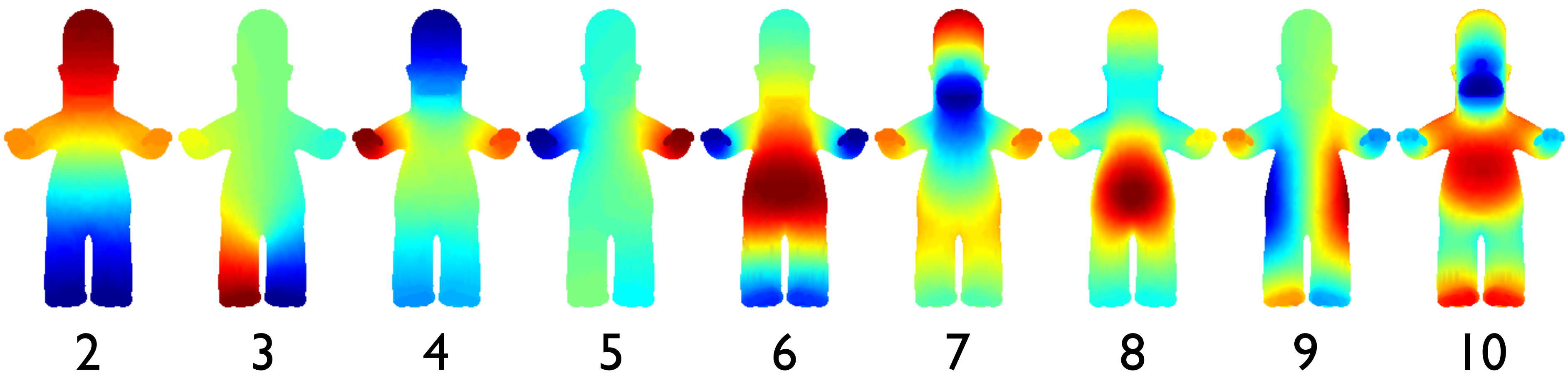
http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Intrinsic Observation

Heat diffusion patterns are not affected if you bend a surface.

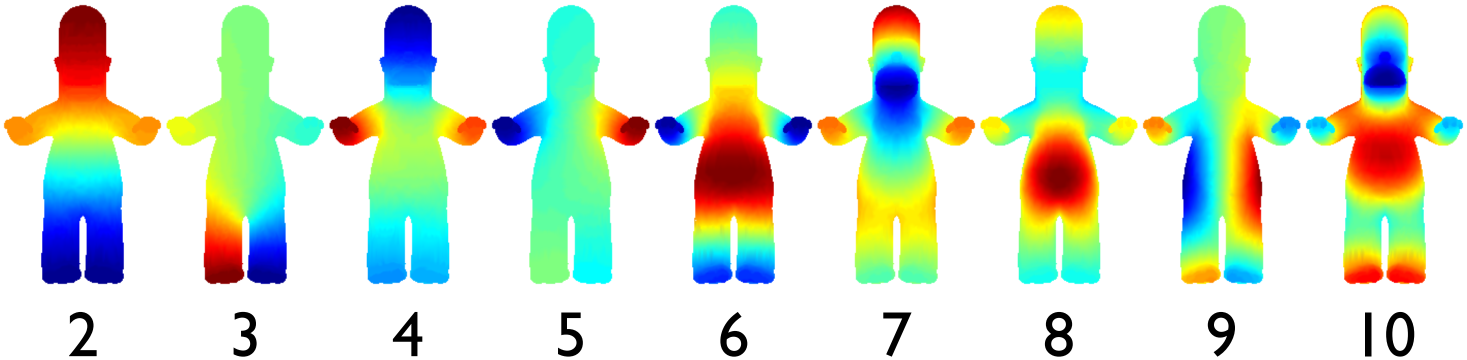
Global Point Signature



$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

"Laplace-Beltrami Eigenfunctions for Deformation Invariant Shape Representation"
Rustamov, SGP 2007

Global Point Signature

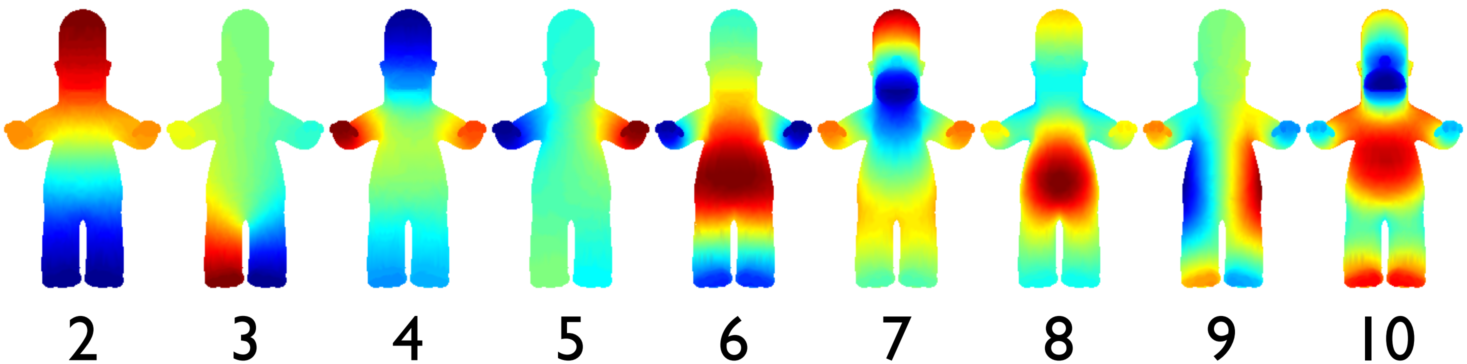


$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

If surface does not self-intersect, neither does the GPS embedding.

Proof: Laplacian eigenfunctions span ; if GPS(p)=GPS(q), then all functions on would be equal at p and q.

Global Point Signature

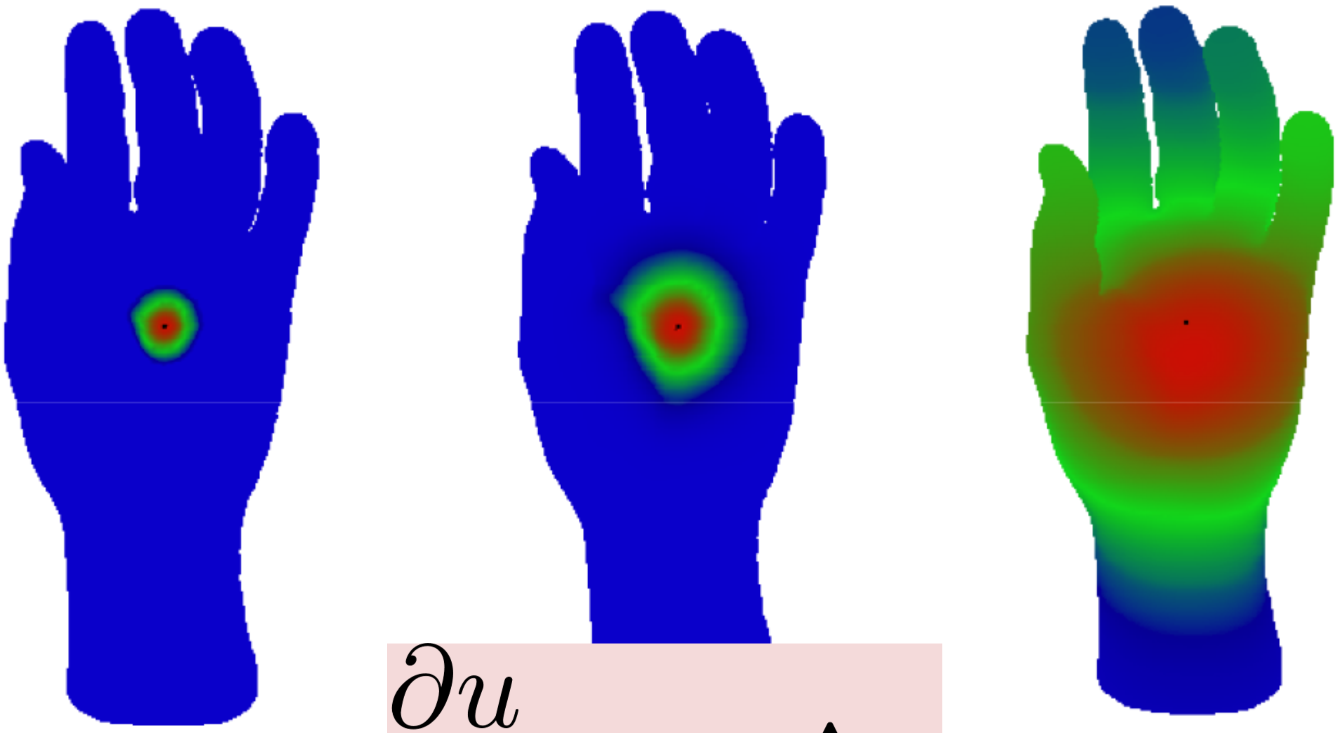


$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

GPS is isometry-invariant.

Proof: Comes from the Laplacian.

Recall: Connection to Physics

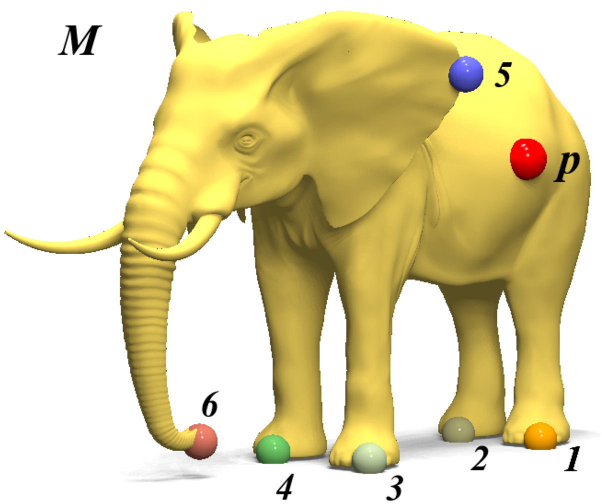


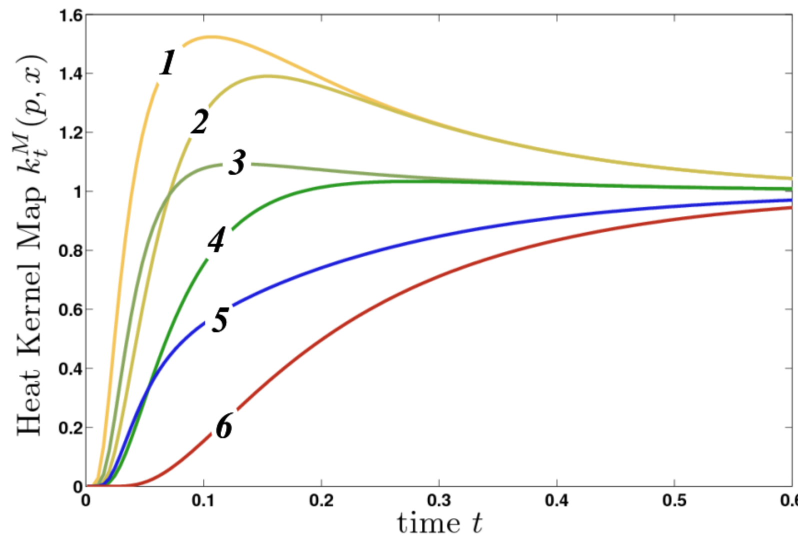
$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Heat Kernel Map



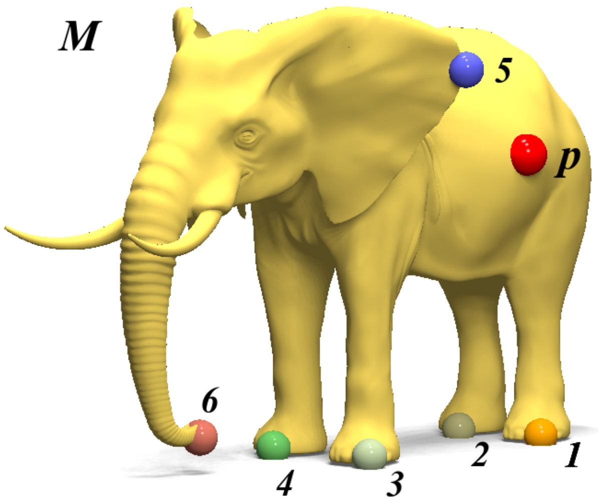


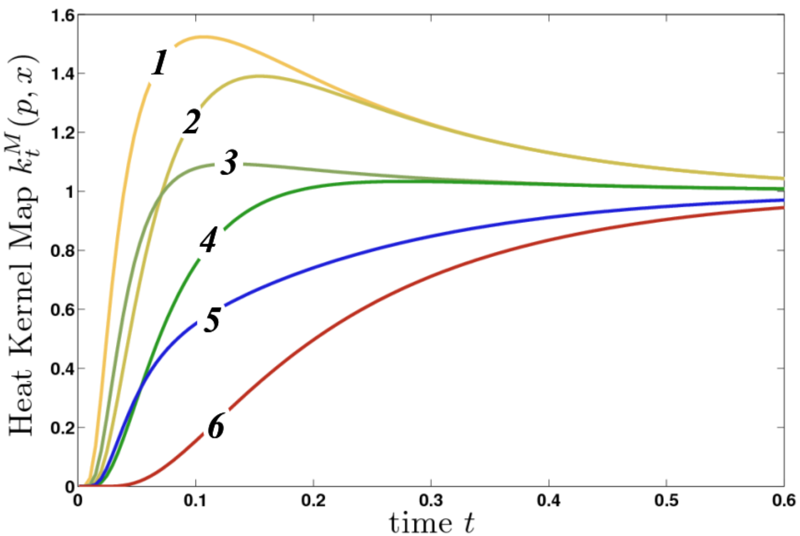
$$\mathrm{HKM}_p(x, t) := k_t(p, x)$$

How much heat diffuses from *p* to *x* in time *t*?

One Point Isometric Matching with the Heat Kernel
Ovsjanikov et al. 2010

Heat Kernel Map





$$\mathrm{HKM}_p(x,t) := k_t(p,x)$$

Theorem: Only have to match one point!

One Point Isometric Matching with the Heat Kernel
Ovsjanikov et al. 2010

KNN