UC San Diego

CSE291-C00

Surfaces

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1. Definition: $\nabla_X f = Xf$

$$Xf = \left(\sum_{i=1}^{2} X^i \frac{\partial}{\partial u_i}\right) f = \sum_{i=1}^{2} X^i \frac{\partial f}{\partial u_i}$$

2. Definition: $\nabla_X \overrightarrow{y} = \nabla_X (y_1, y_2, y_3) = (\nabla_X y_1, \nabla_X y_2, \nabla_X y_3) = (Xf_1, Xf_2, Xf_3)$

3. Definition: $\mathrm{d}Y(X) = \nabla_X \overrightarrow{y}$

   Weingarten map: $\mathrm{d}N(X) = \nabla_X \overrightarrow{n}$

4. Leibniz rule: $\nabla_X \langle \overrightarrow{y}, \vec{z} \rangle = \langle \nabla_X \overrightarrow{y}, \vec{z} \rangle + \langle \overrightarrow{y}, \nabla_X \vec{z} \rangle$

5. $\nabla_X \overrightarrow{y} = \nabla_Y \overrightarrow{x}$ for $\overrightarrow{x} = \mathrm{d}\phi(X)$ and $\overrightarrow{y} = \mathrm{d}\phi(Y)$

   Proof:

$$\nabla_X \overrightarrow{y} = \nabla_X \mathrm{d}\phi(Y) = \sum X^i \frac{\partial}{\partial u^i}\left(\sum \frac{\partial \phi}{\partial u^j} du^j \sum Y^k \frac{\partial}{\partial u^k}\right) = \sum_{i,j} X^i Y^j \frac{\partial^2 \phi}{\partial u^i \partial u^j}$$

6. Second fundamental form: $\mathrm{II}(X, Y) = \langle dN(X), d\phi(Y) \rangle = \langle \nabla_X \overrightarrow{n}, \overrightarrow{y} \rangle$

   Note that $\nabla_X \langle \overrightarrow{n}, \overrightarrow{y} \rangle = \langle \nabla_X \overrightarrow{n}, \overrightarrow{y} \rangle + \langle \overrightarrow{n}, \nabla_X \overrightarrow{y} \rangle$ and $\langle \overrightarrow{n}, \overrightarrow{y} \rangle = 0$

   so $\langle \nabla_X \overrightarrow{n}, \overrightarrow{y} \rangle = -\langle \overrightarrow{n}, \nabla_X \overrightarrow{y} \rangle = -\langle \overrightarrow{n}, \nabla_Y \overrightarrow{x} \rangle = \langle \nabla_Y \overrightarrow{n}, \overrightarrow{x} \rangle$

   $\mathrm{II}(X, Y) = \mathrm{II}(Y, X)$, i.e., the second fundamental form is a symmetric bilinear form

Theorema Egregium

- The Gaussian curvature of a surface can be expressed in terms of the first fundamental form, i.e., it is *intrinsic*

Proof

$$g_{ij,k} = \frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k} \langle \mathbf{x}_i, \mathbf{x}_j \rangle$$

$$\Gamma^\ell_{ij} = \frac{1}{2} g^{k\ell} (g_{ik,j} + g_{jk,i} - g_{ij,k})$$ Christoffel symbols

$$\mathbf{x}_{ij} = \Gamma^\ell_{ij} \mathbf{x}_\ell + L_{ij} n, \qquad (Gauss\ Formulas)$$

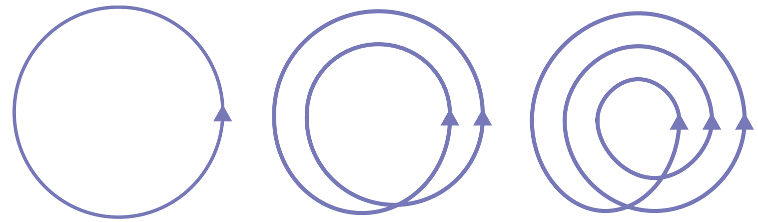
$$\underbrace{\Gamma^\ell_{ik,j} - \Gamma^\ell_{ij,k} + \Gamma^m_{ik}\Gamma^\ell_{mj} - \Gamma^m_{ij}\Gamma^\ell_{mk}}_{R^\ell_{ijk}} = L_{ik} L^\ell{}_j - L_{ij} L^\ell{}_k$$

$$R^\ell_{ijk} = L_{ik} L^\ell{}_j - L_{ij} L^\ell{}_k \qquad The\ Gauss\ Equations$$

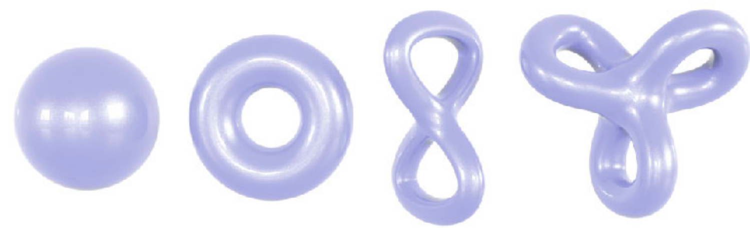
Riemann curvature tensor.

$$K = \frac{g_{2\ell} R^\ell_{121}}{g}$$

- Recall that the total curvature of a closed plane curve was always equal to 2π times *turning number k*
- **Q:** Can we make an analogous statement about surfaces?
- **A:** Yes! Gauss-Bonnet theorem says total Gaussian curvature is always 2π times *Euler characteristic* χ
- Euler characteristic can be expressed in terms of the *genus* (number of "handles")



k=1 k=2 k=3



g=0 g=1 g=2 g=3

Curves

$$\int_0^L \kappa\, ds = 2\pi k$$

Surfaces

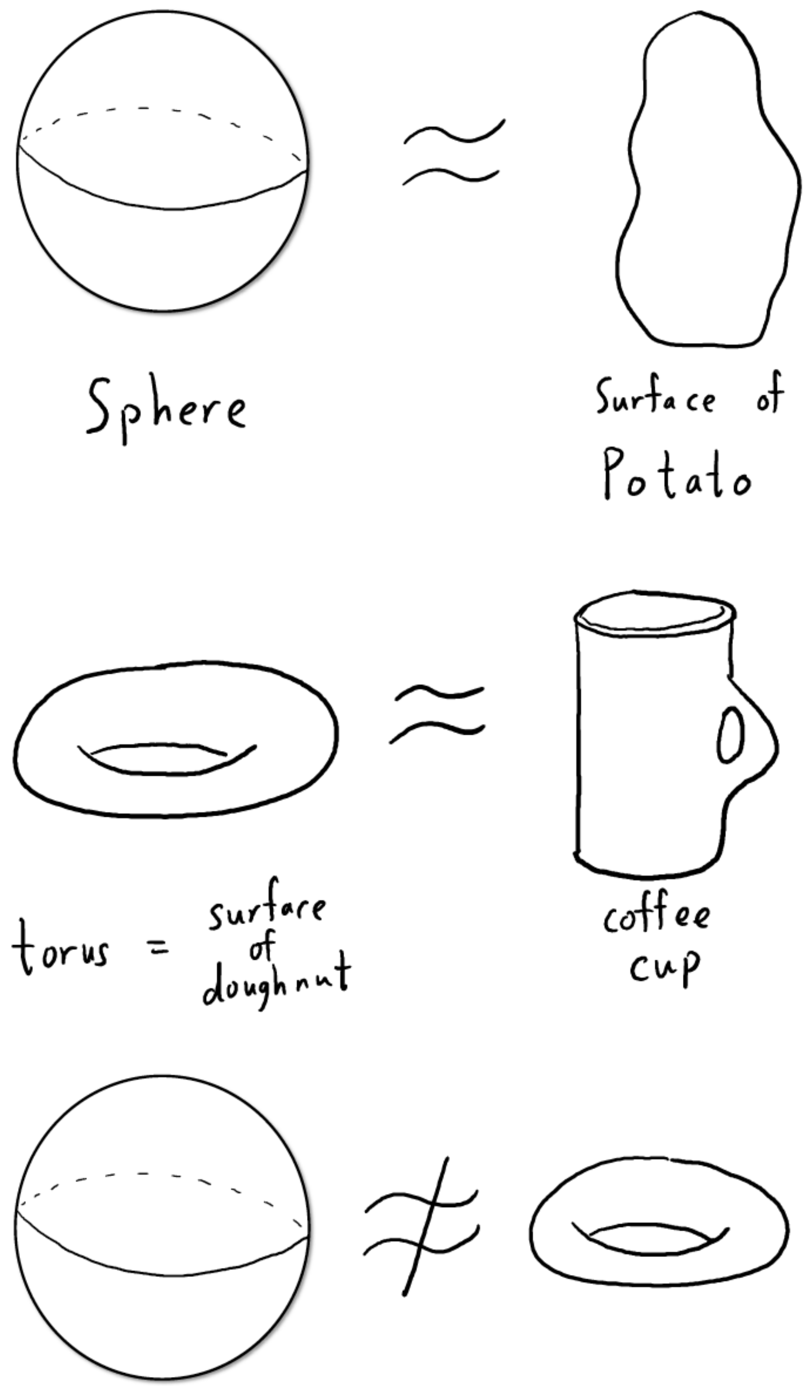
$$\int_M K\, dA = 2\pi\chi$$

Topology of Surfaces

- We will say two surfaces M and N are *topologically equivalent* or are of the same topological type if

M and N are diffeomorphic

- We normally write $M \approx N$ to denote this


Sphere
Surface of Potato
torus = surface of doughnut
coffee cup

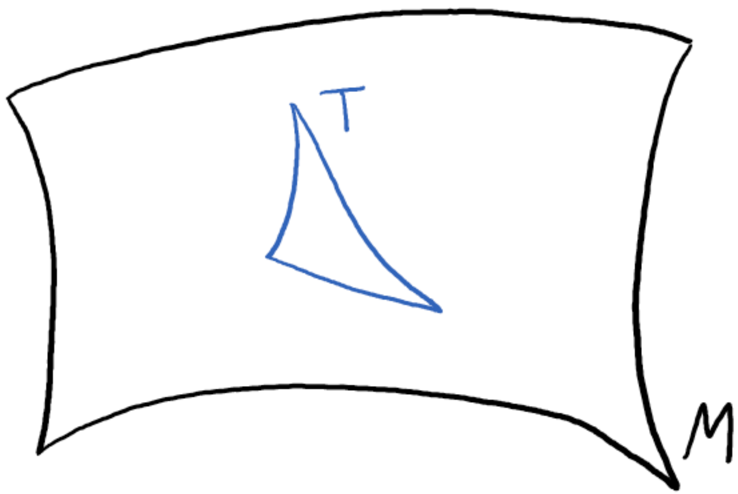
Topology by a Single Number

- It's an astonishing fact that to determine whether two surfaces are topologically equivalent (diffeomorphic) comes down to computing exactly one number of that surface.

- That number is called the **Euler characteristic**.

Triangulation

- A *triangle* T in M is a simple region in M bounded by 3 smooth curve segments.



- Here `simple' means that T is topologically a disk

Triangulation

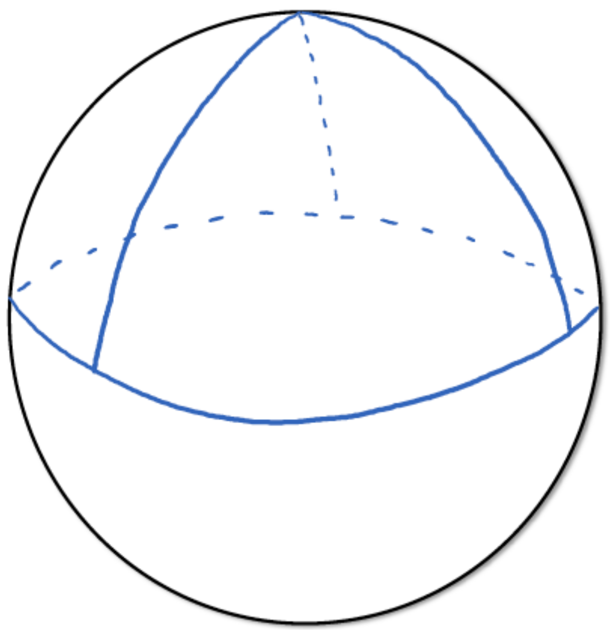
- A *triangulation* of M is a decomposition of M into a finite number of triangles $T_1, T_2, \ldots, T_n$ such that

(1) $\bigcup_{i=1}^{n} T_i = M$

(2) If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common edge or a vertex.

- It's a fact (we will not prove) that every compact surface can be triangulated.

Example. The figure below shows a triangulation of the sphere. Note that the edges of the triangles are great circles and hence geodesics. The triangulation has the same topology type as a tetrahedron. The number of faces is $F = 4$, the number of edges is $E = 6$, and the number of vertices is $V = 4$.



Euler Characteristic

Definition. Let M be a compact surface and consider any triangulation of M. Then the *Euler characteristic* of M is

$$\chi(M) = V - E + F$$

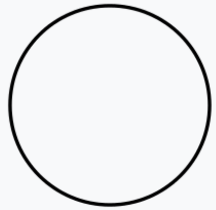
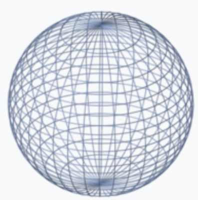
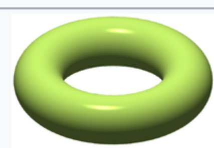
where

$$\begin{aligned} V &= \text{number of vertices} \\ E &= \text{number of edges} \\ F &= \text{number of faces} \end{aligned}$$

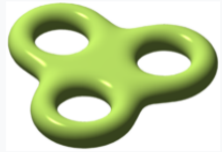
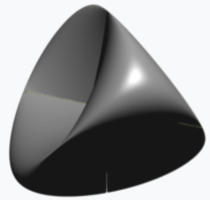
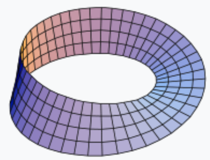
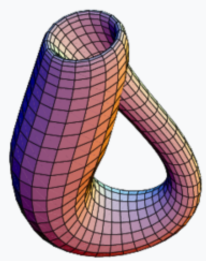
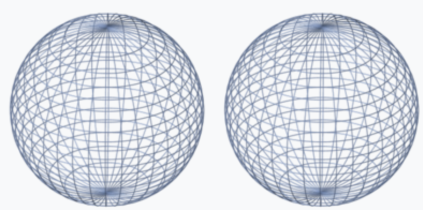
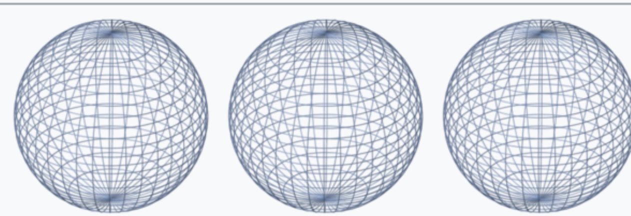
The following fact (we will not prove) justifies the definition of $\chi(M)$.

Theorem 6.9. *The Euler characteristic $\chi(M)$ does not depend on the particular triangulation of M.*

Examples

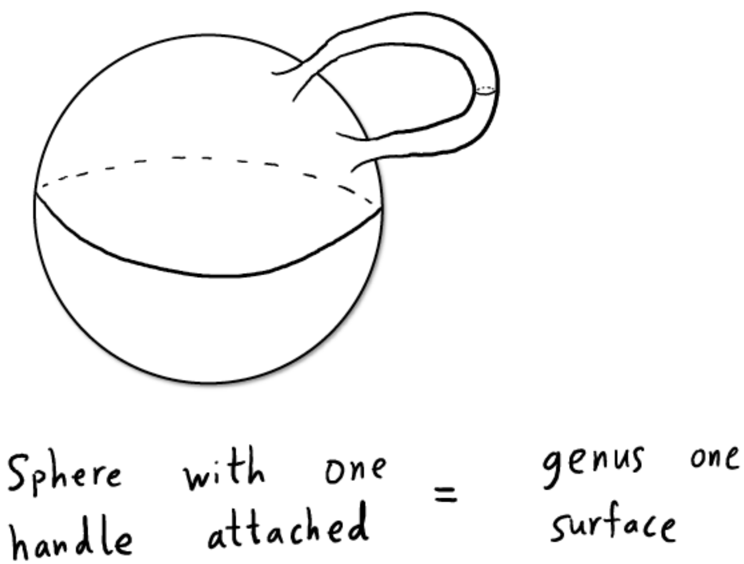
Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		–2

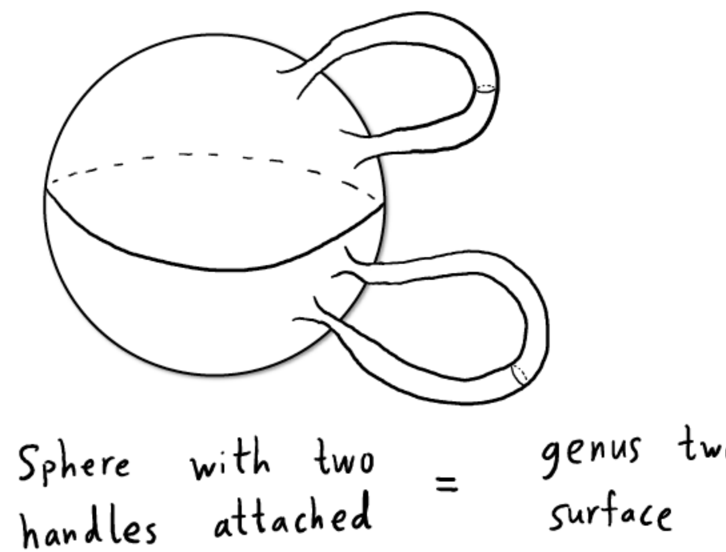
Examples

Triple torus		**−4**
Real projective plane		**1**
Möbius strip		**0**
Klein bottle		**0**
Two spheres (not connected) (Disjoint union of two spheres)		2 + 2 = **4**
Three spheres (not connected) (Disjoint union of three spheres)		2 + 2 + 2 = **6**

Genus

- There is an easy way to construct surfaces with different topology. The idea is to `glue' handles onto a sphere.





Definition. When we construct a surface M in this way with g handles, then we say M is a *surface of genus* g.

Genus and Euler characteristics

Proposition 6.11. *If M is a surface of genus g, then $\chi(M) = 2(1-g)$.*

The proof follows from a formula involving the *connected sum* of two surfaces: $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

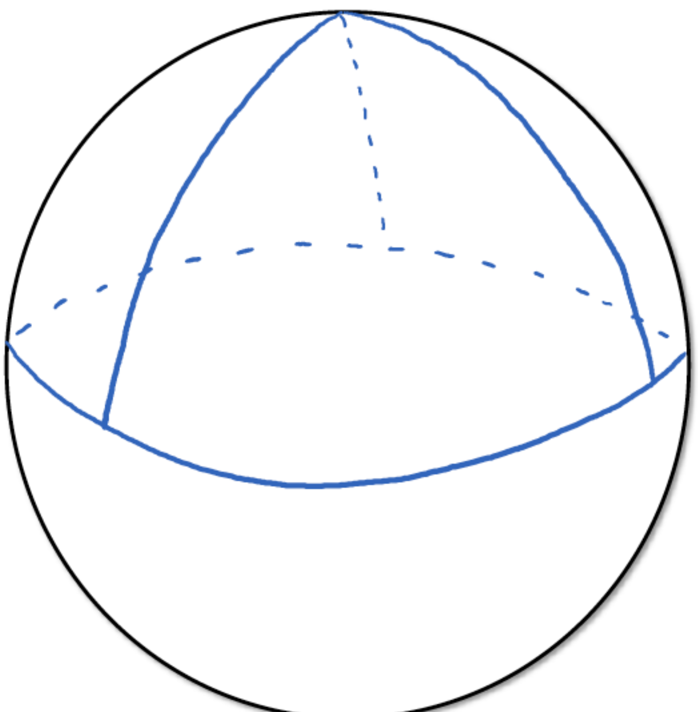
A more general relationship in high-dimensional space:

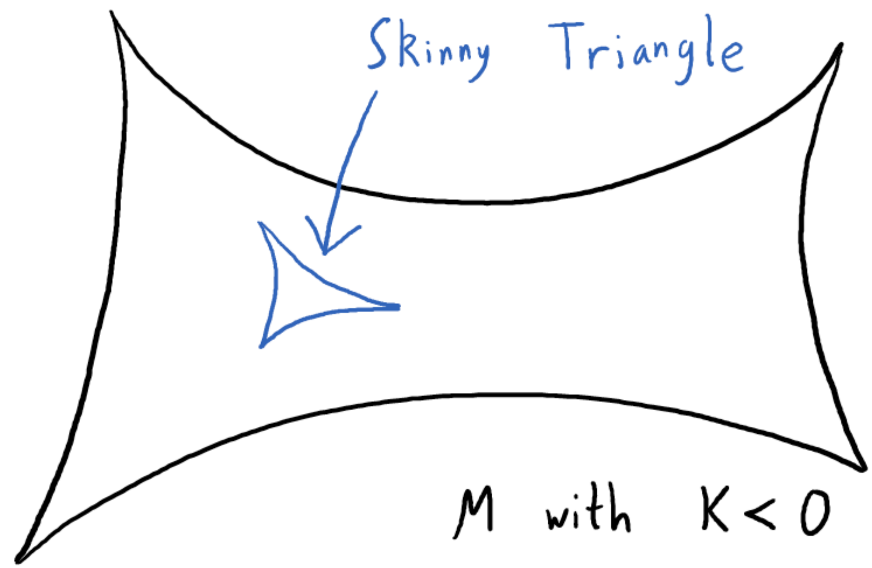
$$\chi(M \# N) = \chi(M) + \chi(N) - \chi(S^n)$$

Main Tool: Angel Excess Theorem

Theorem 6.15 (Angle Excess Theorem). *Let T be a geodesic triangle with angles A, B, and C. Then*

$$A + B + C = \pi + \iint_T K dS.$$





Proof

First, triangulate the surface to n triangles, then

$$\iint_M K dS = \sum_{i=1}^{n} \iint_{T_i} K dS = \sum_{i=1}^{n} (A_i + B_i + C_i - \pi) = \sum_{i=1}^{n} (A_i + B_i + C_i) - n\pi.$$

Proof

First, triangulate the surface to n triangles, then

$$\iint_M K dS = \sum_{i=1}^{n} \iint_{T_i} K dS = \sum_{i=1}^{n} (A_i + B_i + C_i - \pi) = \sum_{i=1}^{n} (A_i + B_i + C_i) - n\pi.$$

So

$$\iint_M K dS = 2\pi V - n\pi = 2\pi V - \pi F$$

Proof

First, triangulate the surface to n triangles, then

$$\iint_M K dS = \sum_{i=1}^{n} \iint_{T_i} K dS = \sum_{i=1}^{n} (A_i + B_i + C_i - \pi) = \sum_{i=1}^{n} (A_i + B_i + C_i) - n\pi.$$

So

$$\iint_M K dS = 2\pi V - n\pi = 2\pi V - \pi F$$

but $E = \frac{3}{2} F$

Proof

First, triangulate the surface to n triangles, then

$$\iint_M K dS = \sum_{i=1}^{n} \iint_{T_i} K dS = \sum_{i=1}^{n} (A_i + B_i + C_i - \pi) = \sum_{i=1}^{n} (A_i + B_i + C_i) - n\pi.$$

So

$$\iint_M K dS = 2\pi V - n\pi = 2\pi V - \pi F$$

but $E = \frac{3}{2}F$

So

$$\iint_M K dS = 2\pi V - \pi F = 2\pi \left(V + F - \frac{3}{2}F \right) = 2\pi(V + F - E) = 2\pi\chi(M).$$