UC San Diego

Lecture 6:

Geometry Foundations (II)

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Agenda

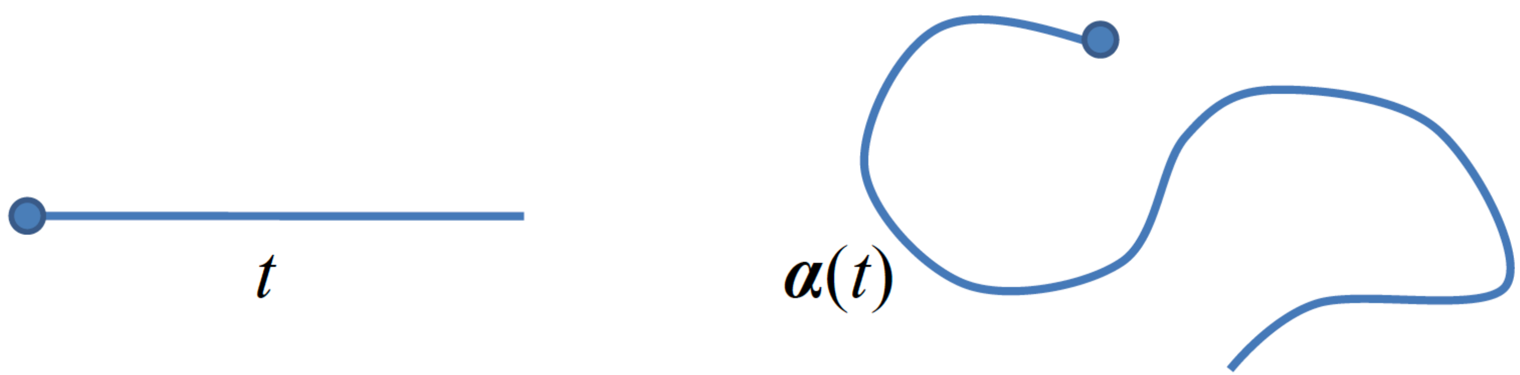
- Curve
- Surface
- Introduction of Geometry Processing

Parameterized Curves
Intuition

A particle is moving in space (E^2, E^3)

At time t its position is given by

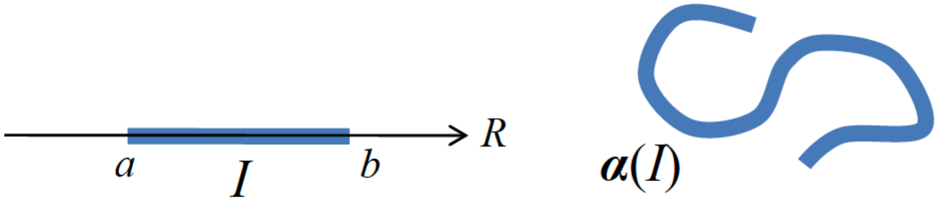
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$$



Parameterized Curves

Definition

A *parameterized differentiable curve* is a differentiable map $\boldsymbol{\alpha}: I \to R^3$ of an interval $I = (\mathrm{a,b})$ of the real line R into R^3



$\boldsymbol{\alpha}$ maps $t \in I$ into a point $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$ such that $x(t), y(t), z(t)$ are *differentiable*

A function is *differentiable* if it has, at all points, derivatives of all orders

The Tangent Vector

Let

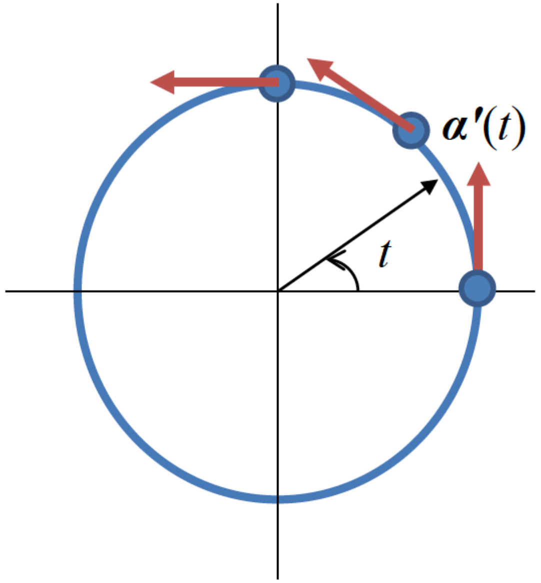
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$$

Then

$$\boldsymbol{\alpha}'(t) = (x'(t), y'(t), z'(t)) \in R^3$$

is called the *tangent vector* (or *velocity vector*) of the curve $\boldsymbol{\alpha}$ at t

Back to the Circle



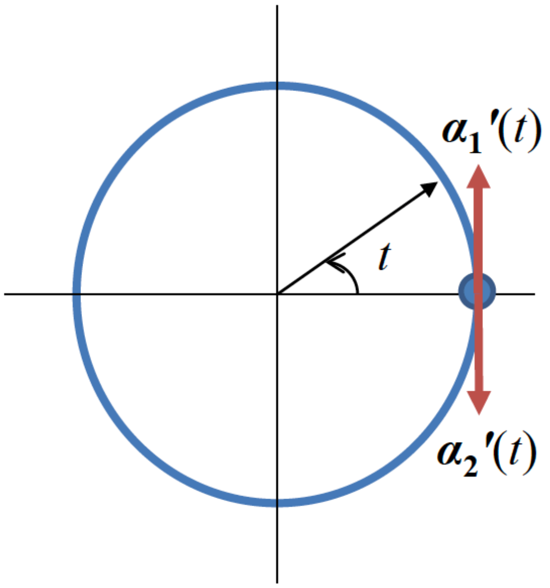
$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t))$$

$$\boldsymbol{\alpha}'(t) = (-\sin(t), \cos(t))$$

$\boldsymbol{\alpha}'(t)$ - direction of movement

$|\boldsymbol{\alpha}'(t)|$ - speed of movement

Back to the Circle



$$\alpha_1(t) = (\cos(t), \sin(t))$$

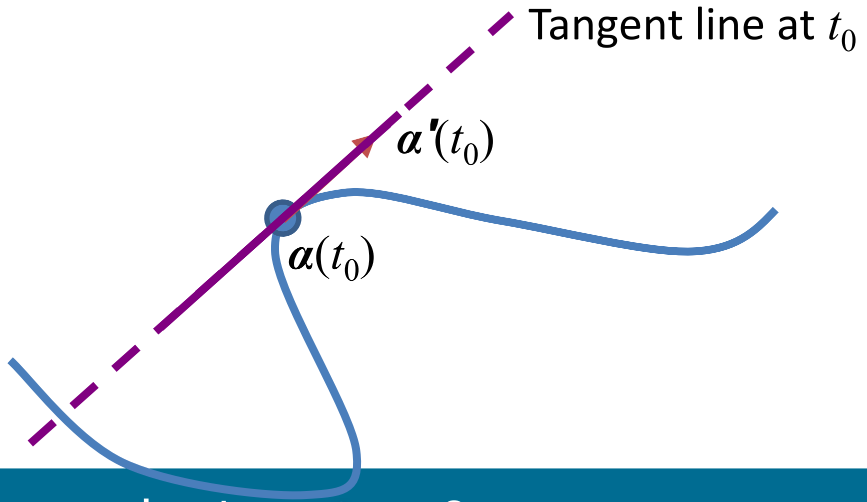
$$\alpha_2(t) = (\cos(-t), \sin(-t))$$

Same speed, different direction

The Tangent Line

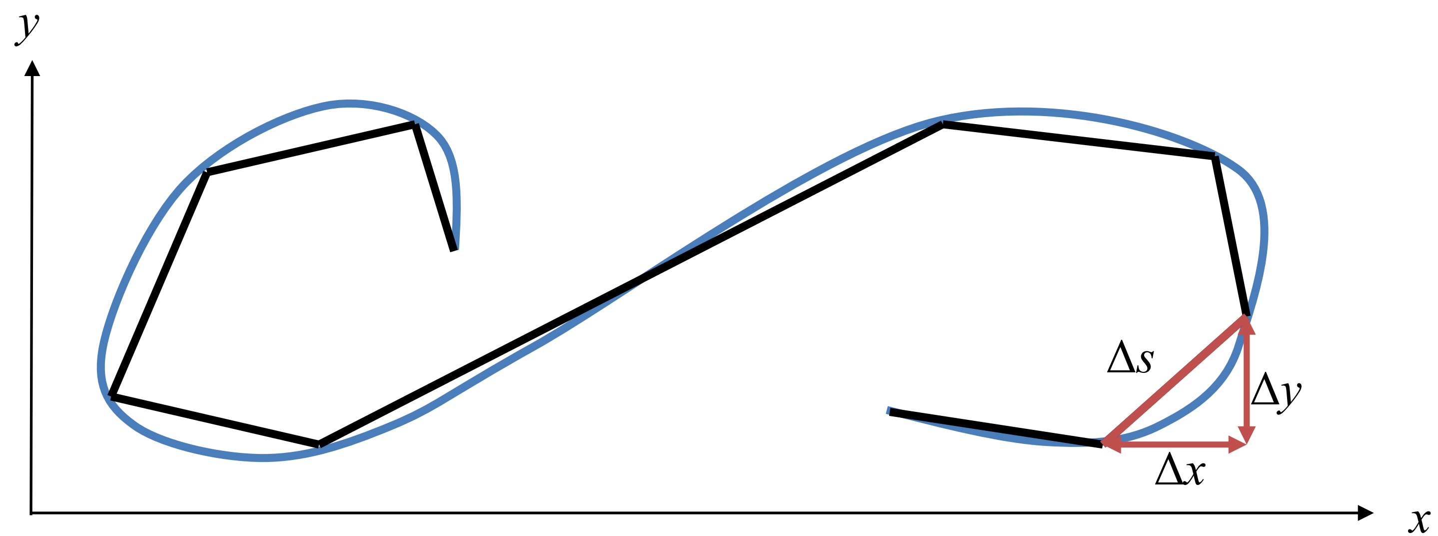
Let $\boldsymbol{\alpha}: I \rightarrow R^3$ be a parameterized differentiable curve.

For each $t \in I$ s.t. $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ the *tangent line* to $\boldsymbol{\alpha}$ at t is the line which contains the point $\boldsymbol{\alpha}(t)$ and the vector $\boldsymbol{\alpha}'(t)$



Arc Length of a Curve

How long is this curve?



Approximate with straight lines

Sum lengths of lines: $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Arc Length

Let $\boldsymbol{\alpha}: I \rightarrow R^3$ be a parameterized differentiable curve. The *arc length* of $\boldsymbol{\alpha}$ from the point t_1 is:

$$s(t) = \int_{t_1}^{t} |\alpha'(p)|\, dp$$

$$= \int_{t_1}^{t} \sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2}\, dp$$

The arc length is an *intrinsic* property of the curve – does not depend on choice of parameterization

Arc Length Parameterization

A curve $\alpha: I \rightarrow R^3$ is *parameterized by arc length* if $|\alpha'(t)| = 1$, for all t

For such curves we have

$$s(t) = \int_{t_0}^{t} dt' = t - t_0$$

The Local Theory of Curves

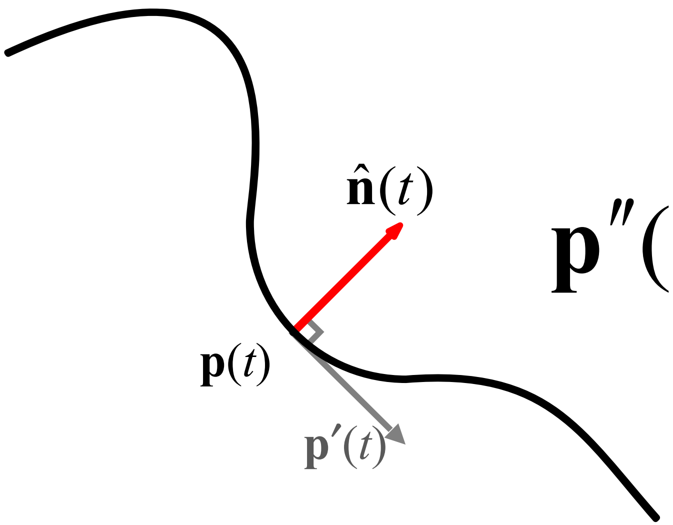
Defines local properties of curves

Local = properties which depend only on behavior in neighborhood of point

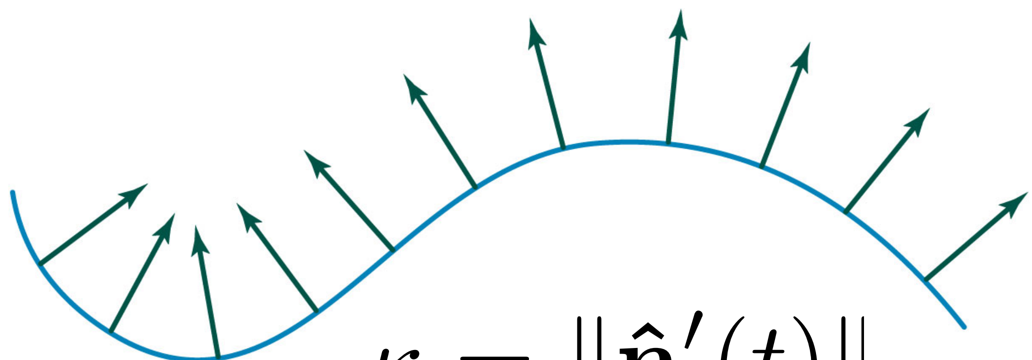
We will consider only curves parameterized by arc length

Curvature and Normal

- Assuming t is arc-length parameter:



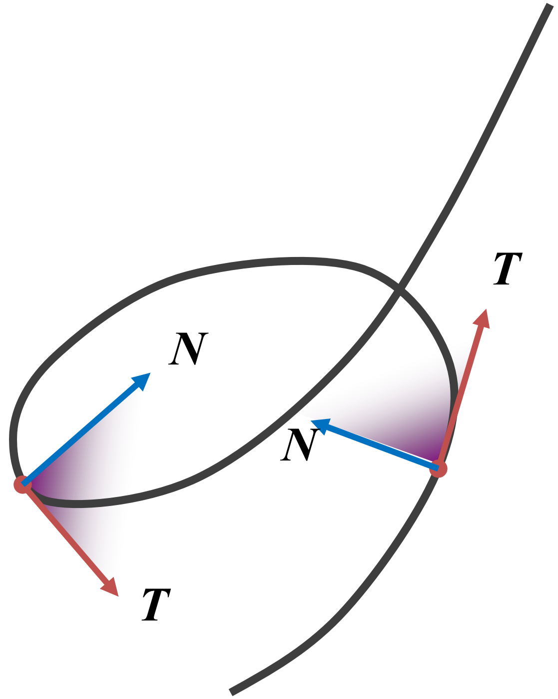
$$\mathbf{p}''(t) = \kappa \hat{\mathbf{n}}(t)$$ normal to the curve



$$\kappa = \|\hat{\mathbf{n}}'(t)\|$$

The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s

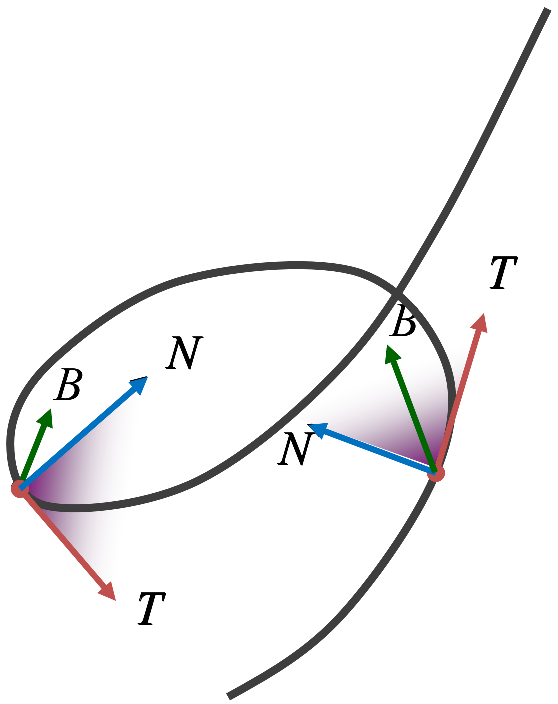


The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

The binormal vector defines the osculating plane

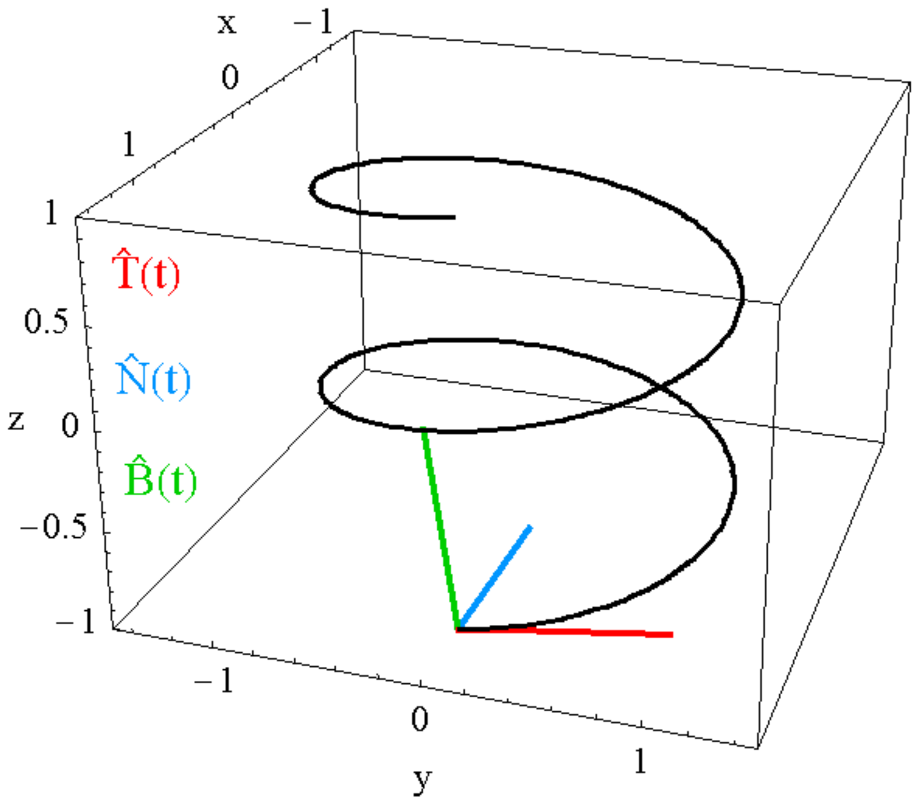


The Frenet Frame

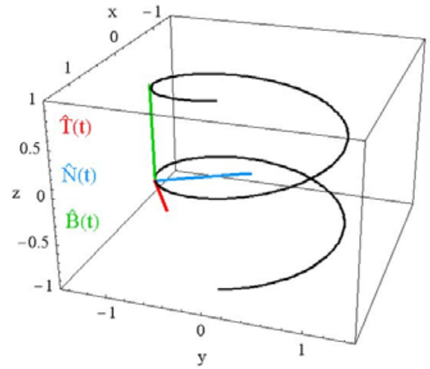
$\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ form an orthonormal basis for R^3 called the *Frenet frame*

How does the frame change when the particle moves?

What are $\boldsymbol{T}', \boldsymbol{N}', \boldsymbol{B}'$ in terms of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$?



$T'(s)$

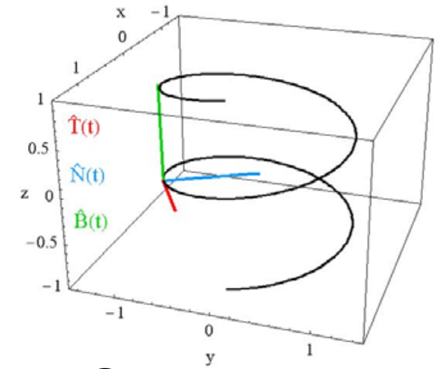


Already used it to define the curvature:

$$\boxed{T'(s) = \kappa(s)N(s)}$$

Since in the direction of the normal, its orthogonal to B and T

$\boldsymbol{N}'(s)$



What is $\boldsymbol{N}'(s)$ as a combination of $\boldsymbol{N}, \boldsymbol{T}, \boldsymbol{B}$?

We know: $\boldsymbol{N}(s) \cdot \boldsymbol{N}(s) = 1$

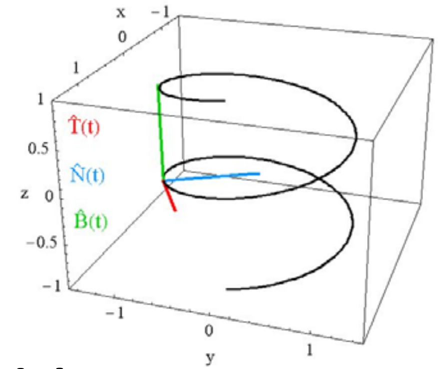
From the lemma $\rightarrow \boldsymbol{N}'(s) \cdot \boldsymbol{N}(s) = 0$

We know: $\boldsymbol{N}(s) \cdot \boldsymbol{T}(s) = 0$

From the lemma $\rightarrow \boldsymbol{N}'(s) \cdot \boldsymbol{T}(s) = -\boldsymbol{N}(s) \cdot \boldsymbol{T}'(s)$

From the definition $\rightarrow \kappa(s) = \boldsymbol{N}(s)\ \boldsymbol{T}'(s)$

$\rightarrow \boldsymbol{N}'(s) \cdot \boldsymbol{T}(s) = -\kappa(s)$



The Torsion

Let $\boldsymbol{\alpha}: I \rightarrow R^3$ be a curve parameterized by arc length s. The *torsion* of $\boldsymbol{\alpha}$ at s is defined by:

$$\tau(s) = \boldsymbol{N}'(s) \cdot \boldsymbol{B}(s)$$

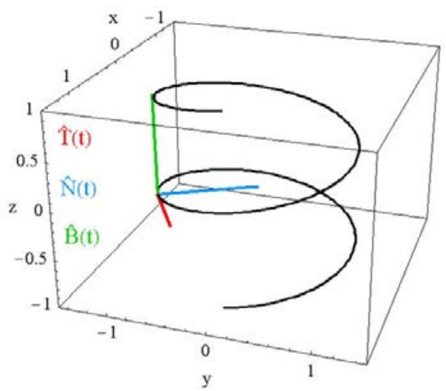
Now we can express $\boldsymbol{N}'(s)$ as:

$$\boxed{\boldsymbol{N}'(s) = -\kappa(s)\,\boldsymbol{T}(s) + \tau(s)\,\boldsymbol{B}(s)}$$

$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s)$

Curvature vs. Torsion

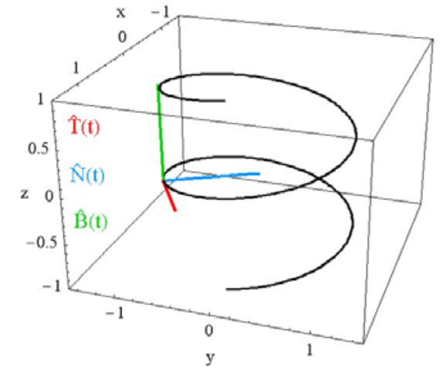
The *curvature* indicates how much the normal changes, in the direction tangent to the curve



The *torsion* indicates how much the normal changes, in the direction orthogonal to the osculating plane of the curve

The curvature is always positive, the torsion can be negative

Both properties *do not* depend on the choice of parameterization



$B'(s)$

What is $\boldsymbol{B}'(s)$ as a combination of $\boldsymbol{N},\boldsymbol{T},\boldsymbol{B}$?

We know: $\boldsymbol{B}(s) \cdot \boldsymbol{B}(s) = 1$

From the lemma $\rightarrow$ $\boldsymbol{B}'(s) \cdot \boldsymbol{B}(s) = 0$

We know: $\boldsymbol{B}(s) \cdot \boldsymbol{T}(s) = 0, \boldsymbol{B}(s) \cdot \boldsymbol{N}(s) = 0$

From the lemma $\rightarrow$

$$\boldsymbol{B}'(s) \cdot \boldsymbol{T}(s) = -\boldsymbol{B}(s) \cdot \boldsymbol{T}'(s) = -\boldsymbol{B}(s) \cdot \kappa(s)\boldsymbol{N}(s) = 0$$

From the lemma $\rightarrow$

$$\boldsymbol{B}'(s) \cdot \boldsymbol{N}(s) = -\boldsymbol{B}(s) \cdot \boldsymbol{N}'(s) = -\tau(s)$$

Now we can express $\boldsymbol{B}'(s)$ as:

$$\boxed{\boldsymbol{B}'(s) = -\tau(s)\,\boldsymbol{N}(s)}$$

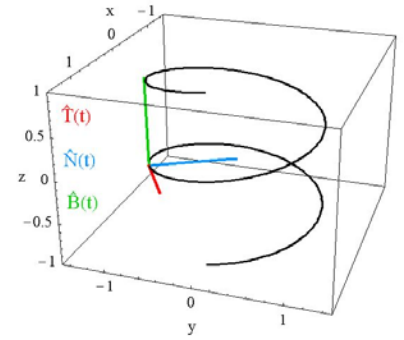
The Frenet Formulas

$$
\begin{aligned}
\boldsymbol{T}'(s) &= \quad\quad\quad \kappa(s)\boldsymbol{N}(s) \\
\boldsymbol{N}'(s) &= -\kappa(s)\boldsymbol{T}(s) \quad\quad\quad + \tau(s)\boldsymbol{B}(s) \\
\boldsymbol{B}'(s) &= \quad\quad\quad -\tau(s)\boldsymbol{N}(s)
\end{aligned}
$$

In matrix form:

$$
\begin{bmatrix} | & | & | \\ \boldsymbol{T}'(s) & \boldsymbol{N}'(s) & \boldsymbol{B}'(s) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \boldsymbol{T}(s) & \boldsymbol{N}(s) & \boldsymbol{B}(s) \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}
$$

An Example – The Helix



$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t), bt)$

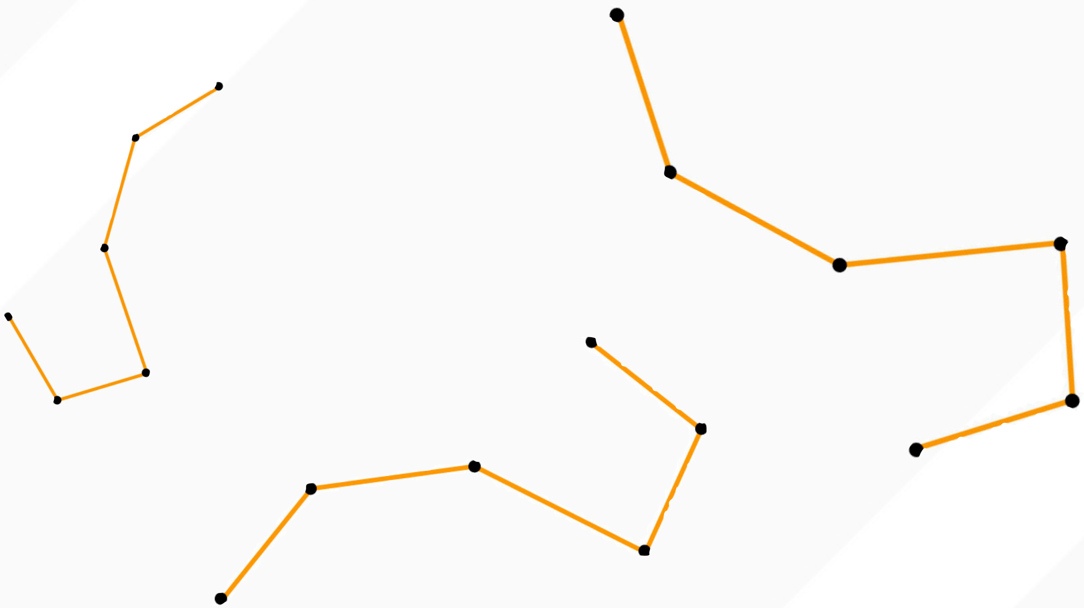
In arc length parameterization:

$\boldsymbol{\alpha}(s) = (a\cos(s/c), a\sin(s/c), bs/c)$, where $c = \sqrt{a^2 + b^2}$

Curvature: $\kappa(s) = \dfrac{a}{a^2 + b^2}$ Torsion: $\tau(s) = \dfrac{b}{a^2 + b^2}$

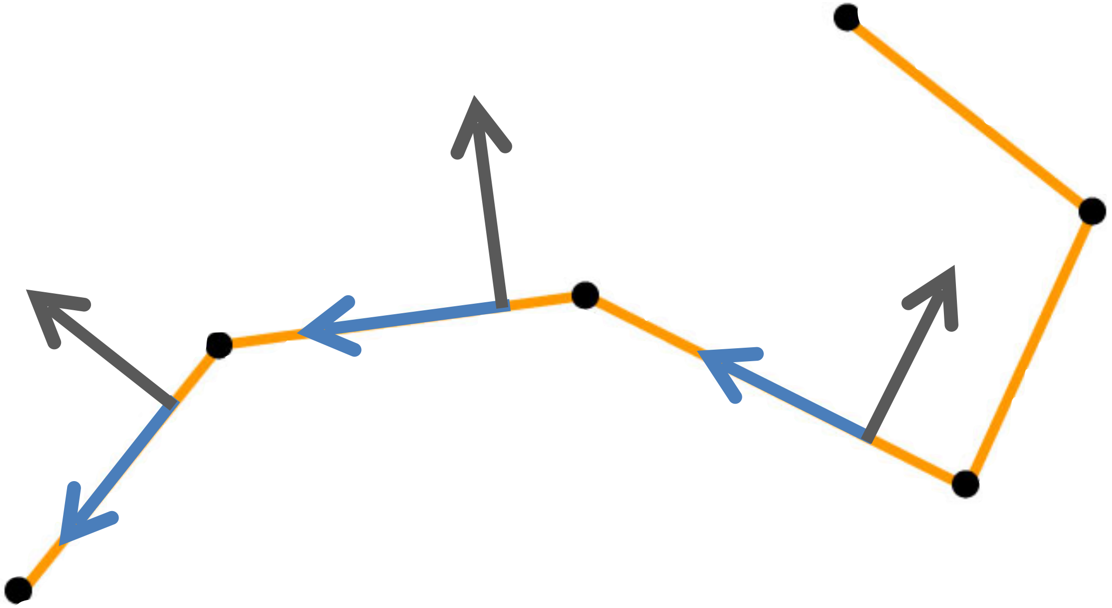
Note that both the curvature and torsion are constants

DISCRETE PLANAR CURVES



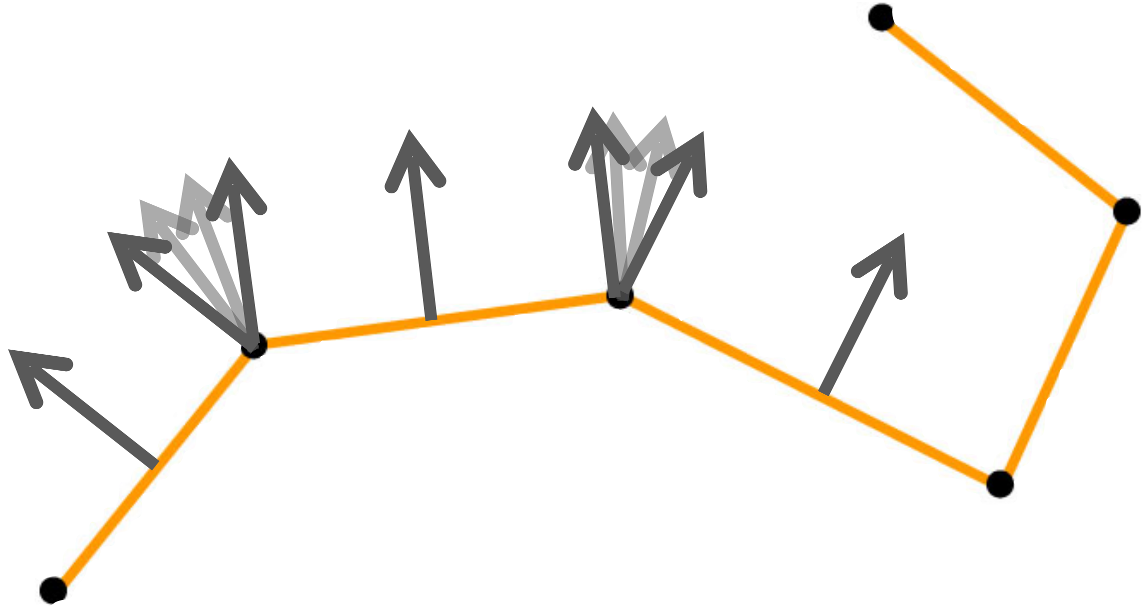
Tangents, Normals

- For any point on the edge, the tangent is simply the unit vector along the edge and the normal is the perpendicular vector



Tangents, Normals

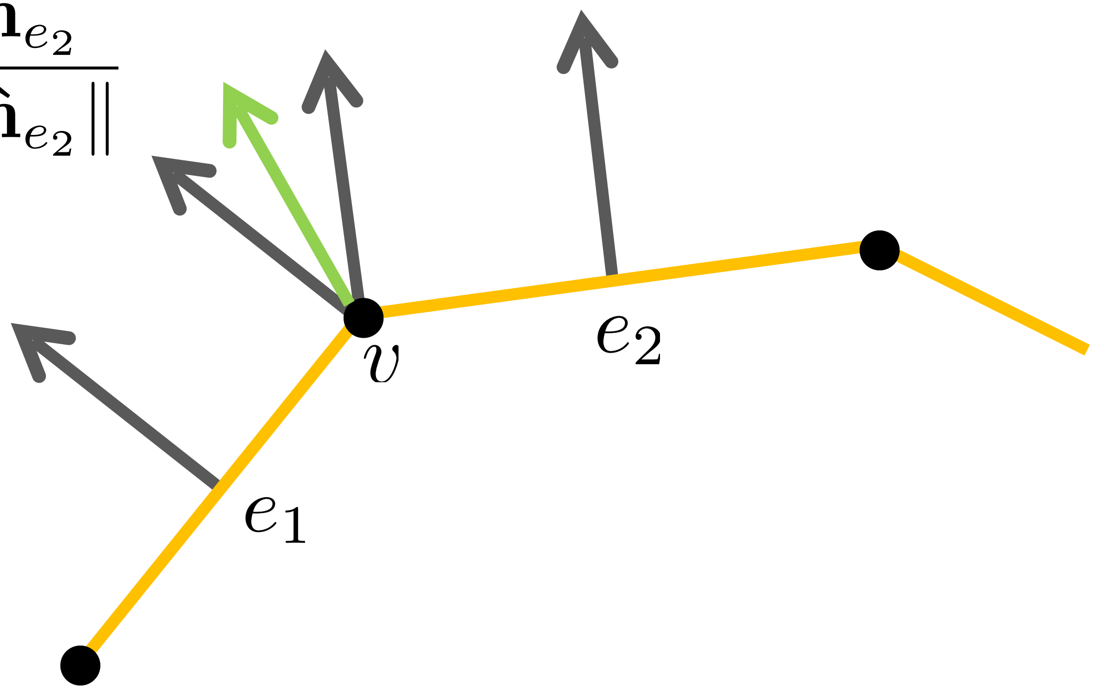
- For vertices, we have many options



Tangents, Normals

- Can choose to average the adjacent edge normals

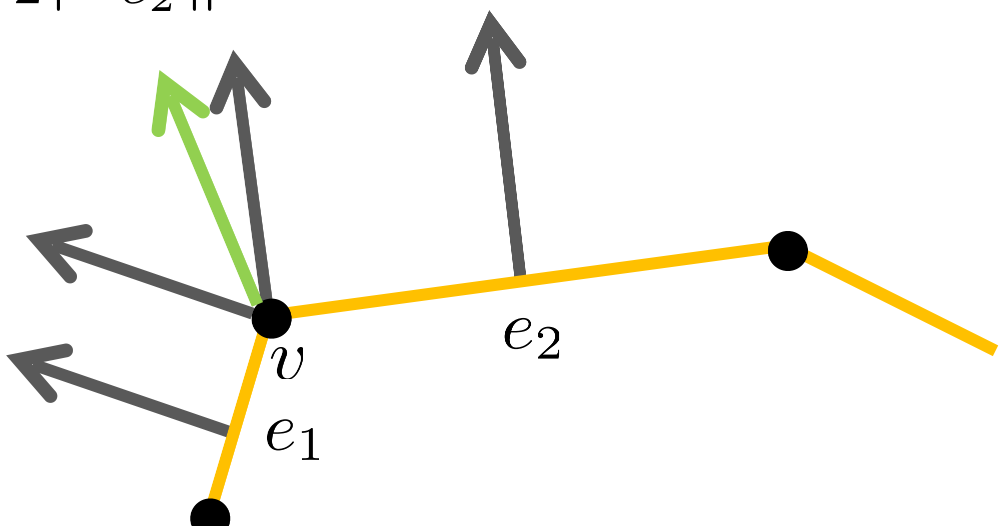
$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

- Weight by edge lengths

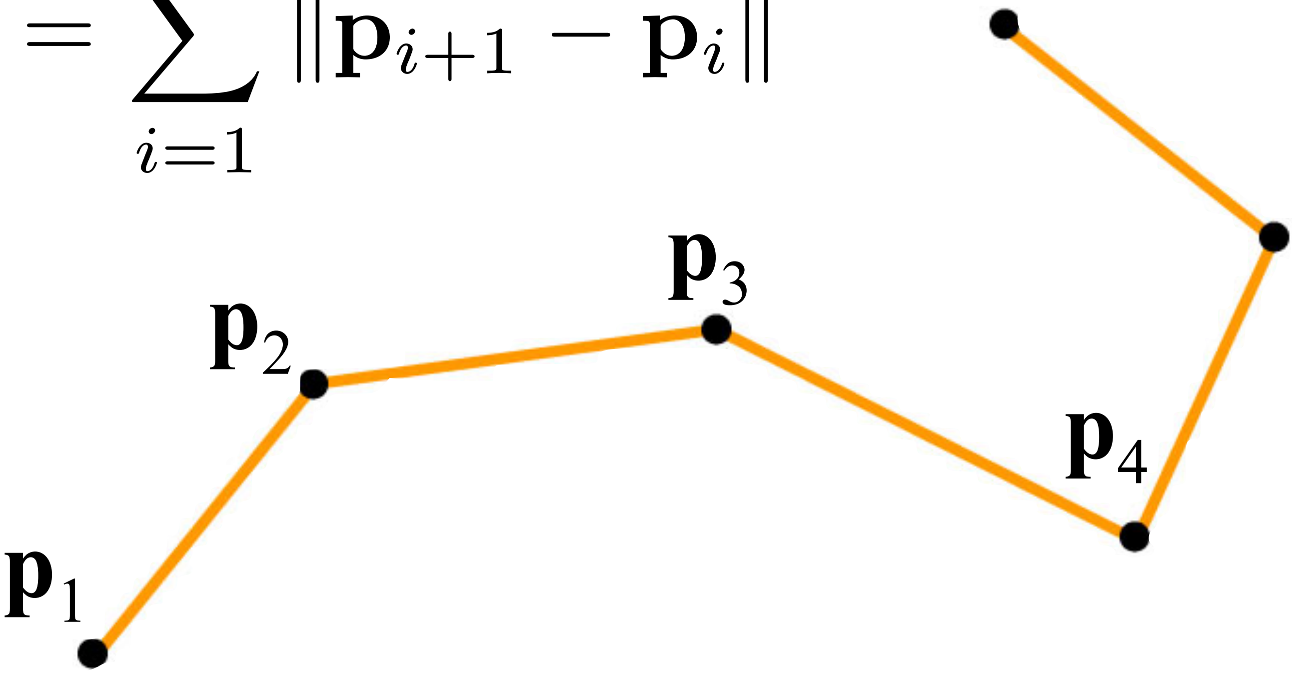
$$\hat{\mathbf{n}}_v = \frac{|e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}}{\||e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}\|}$$



The Length of a Discrete Curve

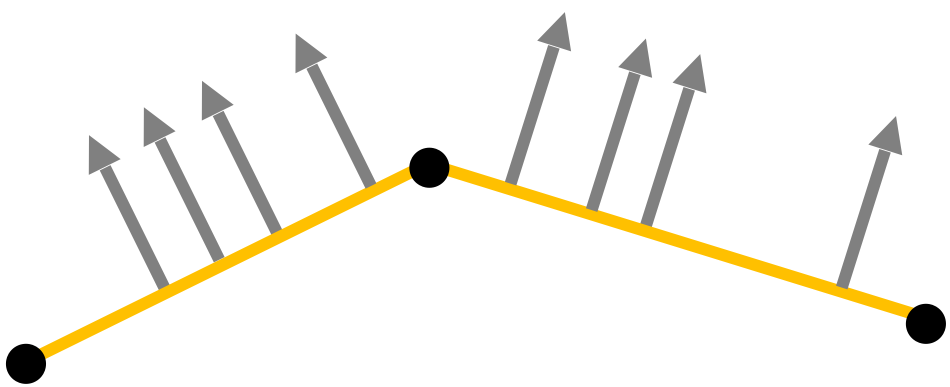
- Sum of edge lengths

$$\mathrm{len}(p) = \sum_{i=1}^{n-1} \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$



Curvature of a Discrete Curve

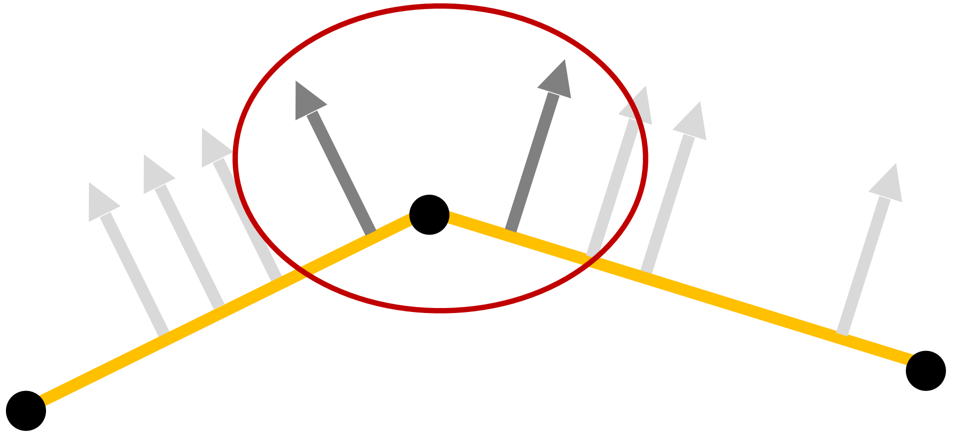
- Curvature is the change in normal direction as we travel along the curve



no change along each edge –
curvature is zero along edges

Curvature of a Discrete Curve

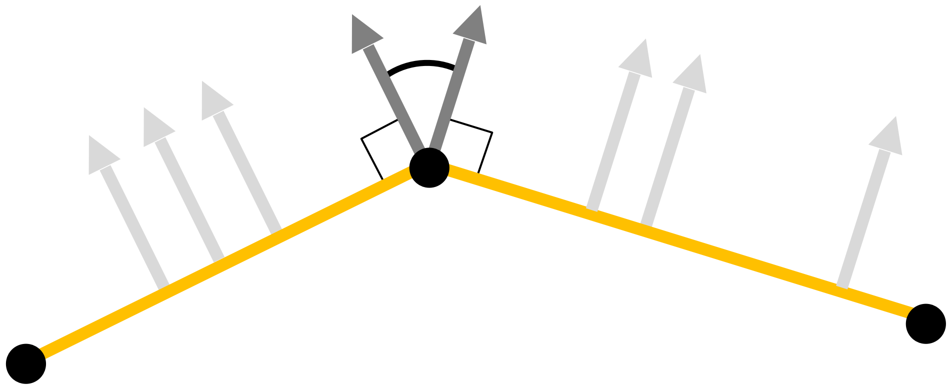
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

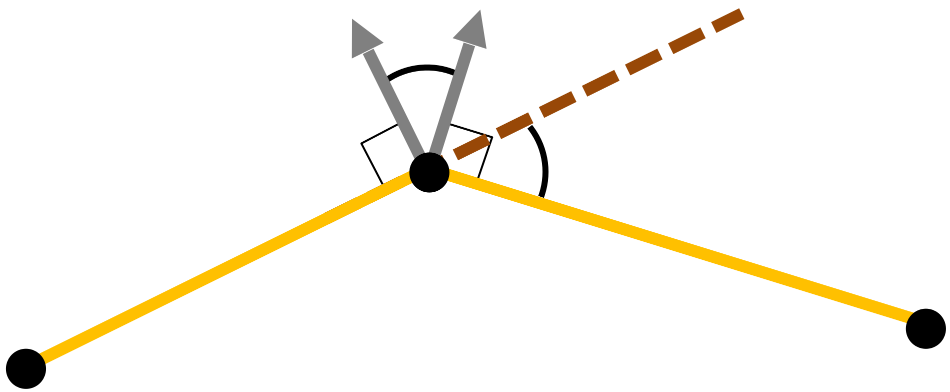
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

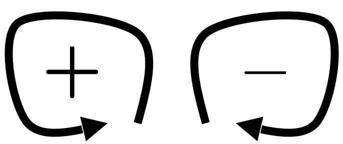
- Curvature is the change in normal direction as we travel along the curve



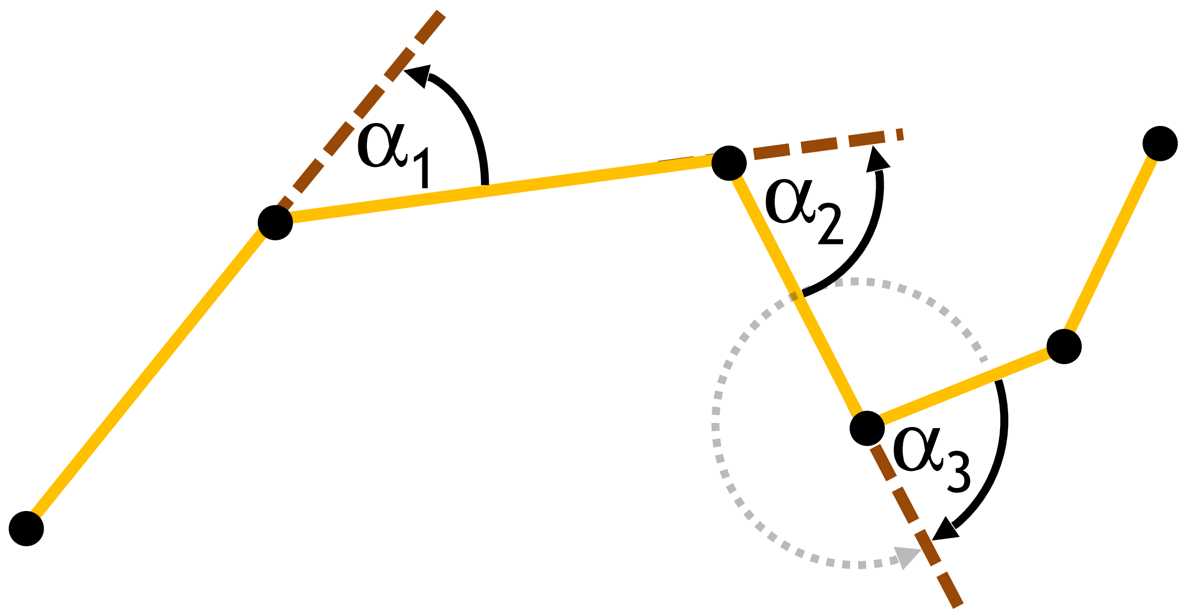
same as the turning angle
between the edges

Curvature of a Discrete Curve

- Zero along the edges
- Turning angle at the vertices
  = the change in normal direction



$\alpha_1,\ \alpha_2 > 0, \quad \alpha_3 < 0$



TURNING NUMBER THEOREM

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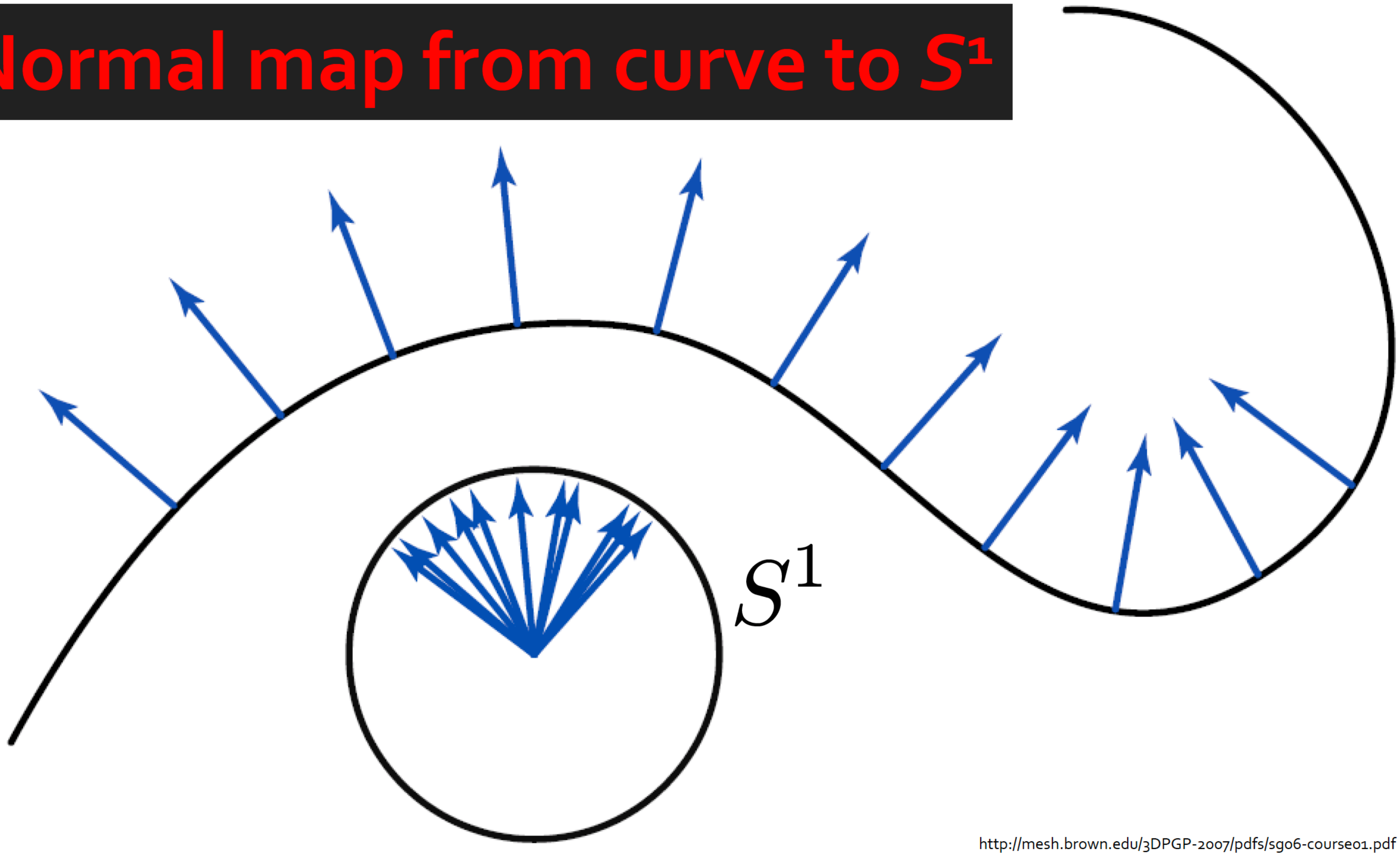
Gauss Map

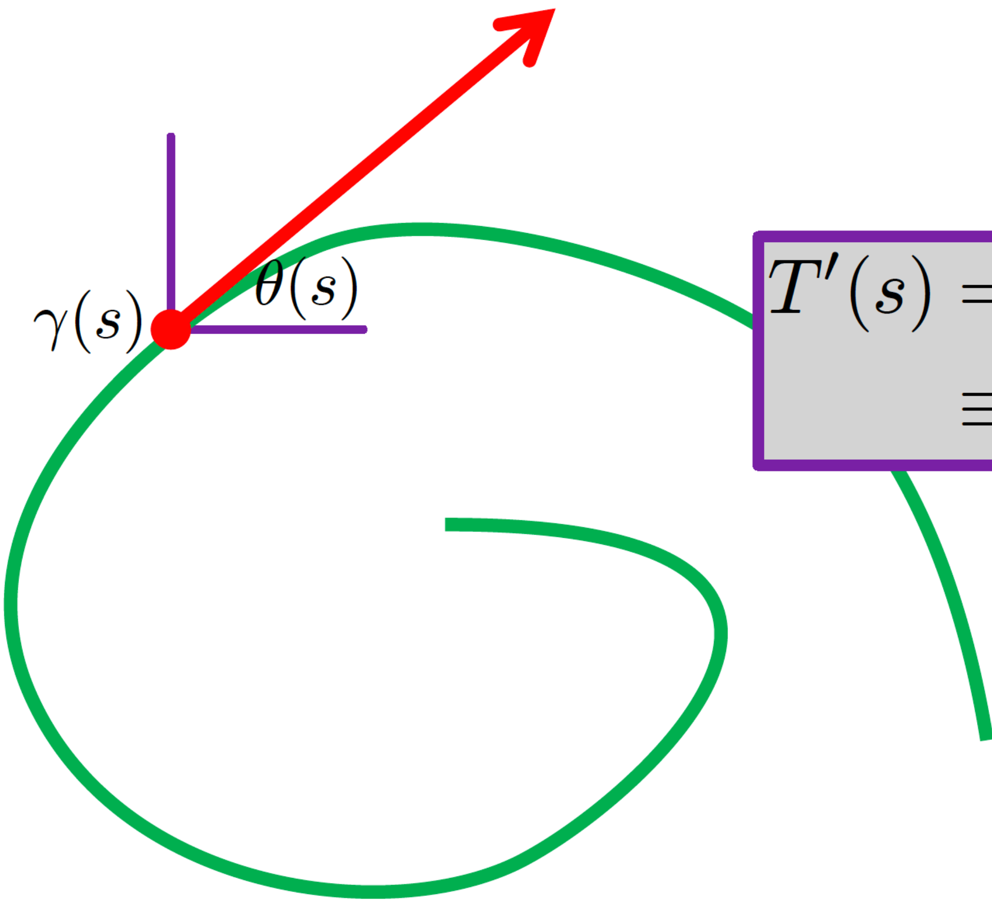
Normal map from curve to S^1



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

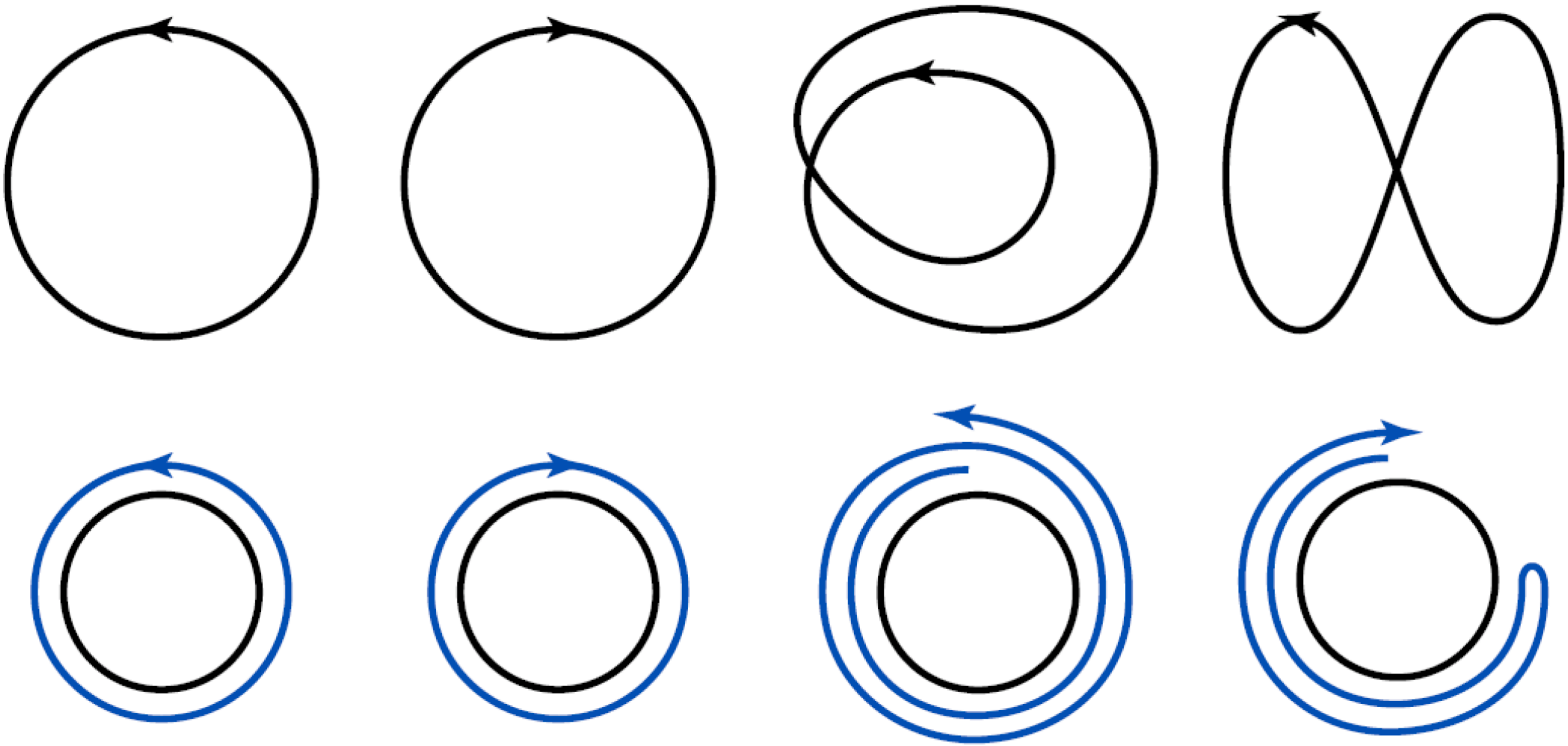
Signed Curvature on Plane Curves

$$T(s) = (\cos\theta(s), \sin\theta(s))$$



$$\boxed{\begin{aligned} T'(s) &= \theta'(s)(-\sin\theta(s), \cos\theta(s)) \\ &\equiv \kappa(s)N(s) \end{aligned}}$$

Turning Numbers



+1 -1 +2 0

http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

Recovering Theta

$$\theta'(s) \equiv \kappa(s)$$

$$\Downarrow$$

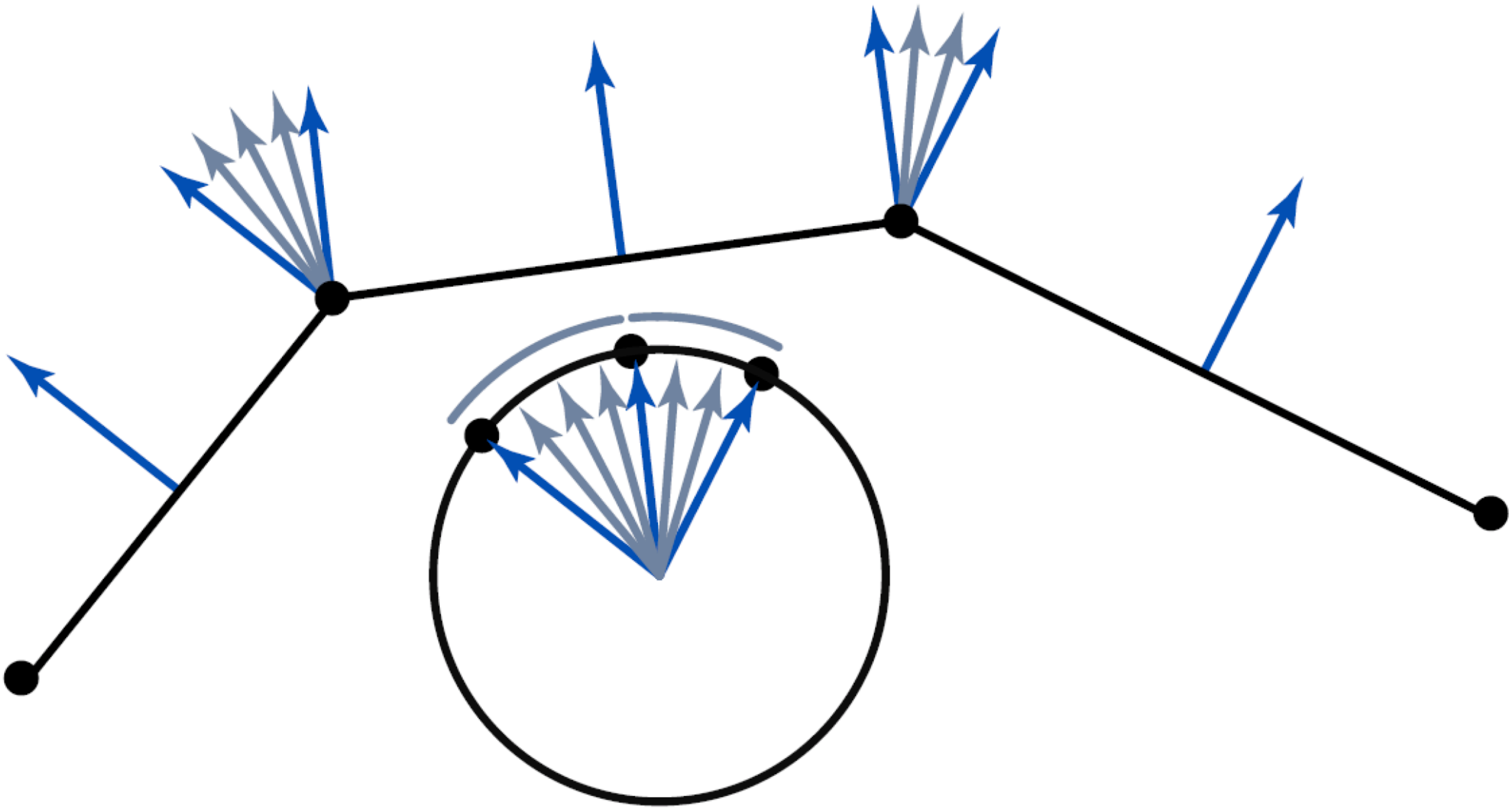
$$\Delta\theta = \int_{s_0}^{s_1} \kappa(s)\ ds$$

Turning Number Theorem

$$\boxed{\int \kappa(s)\ ds = 2\pi k}$$

A "global" theorem!

Discrete Gauss Map



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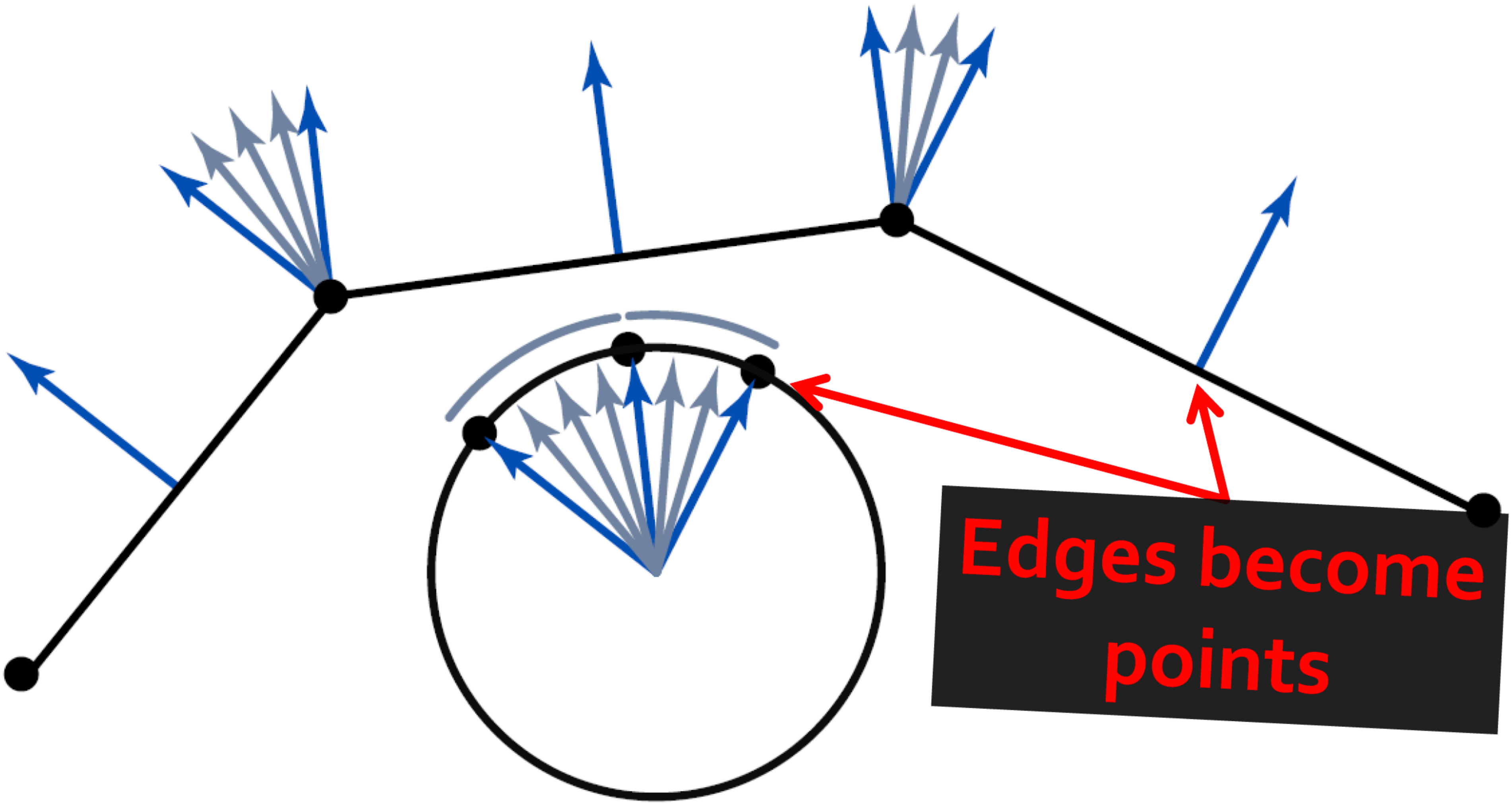
Discrete Gauss Map



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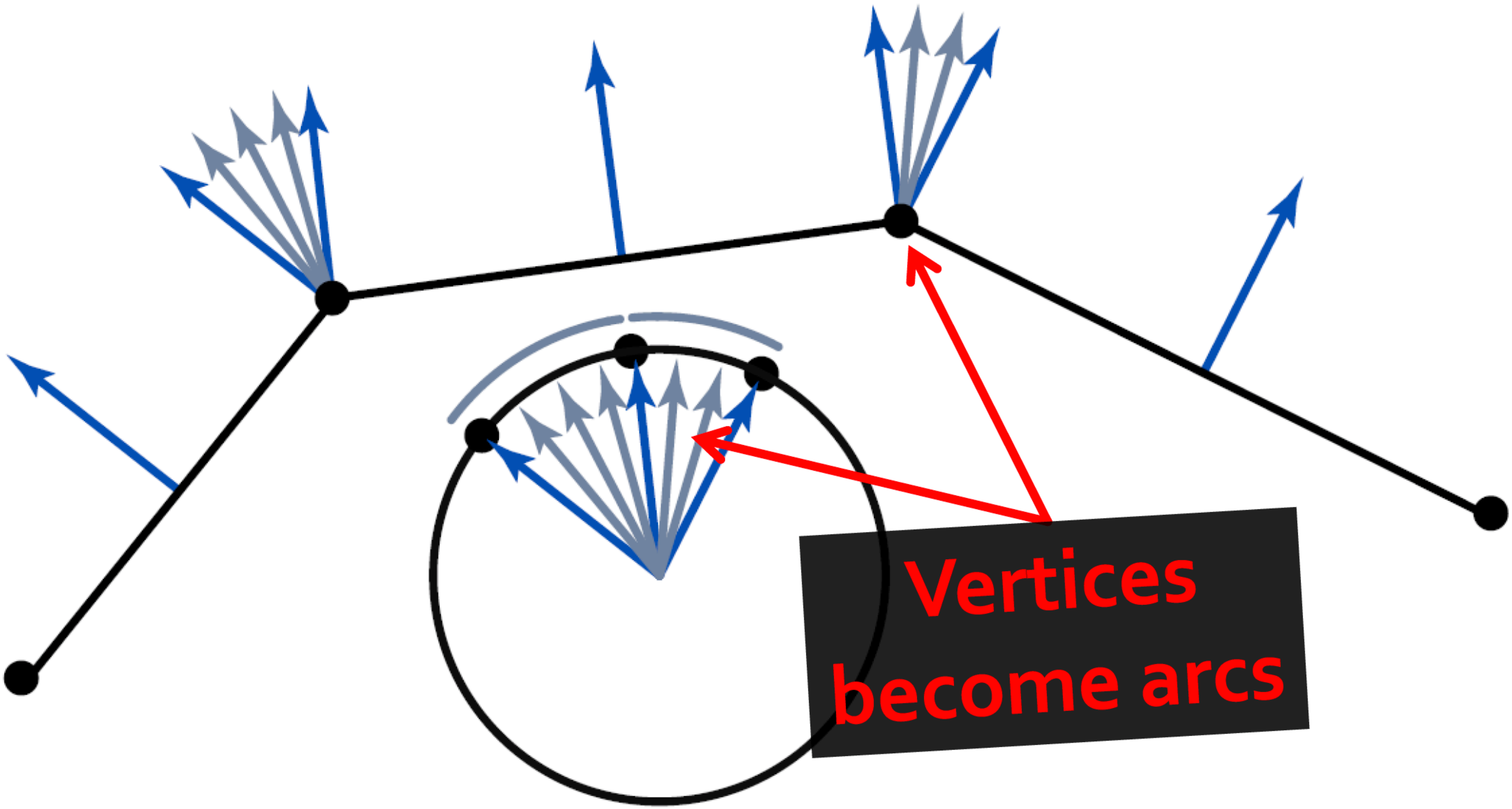
Discrete Gauss Map



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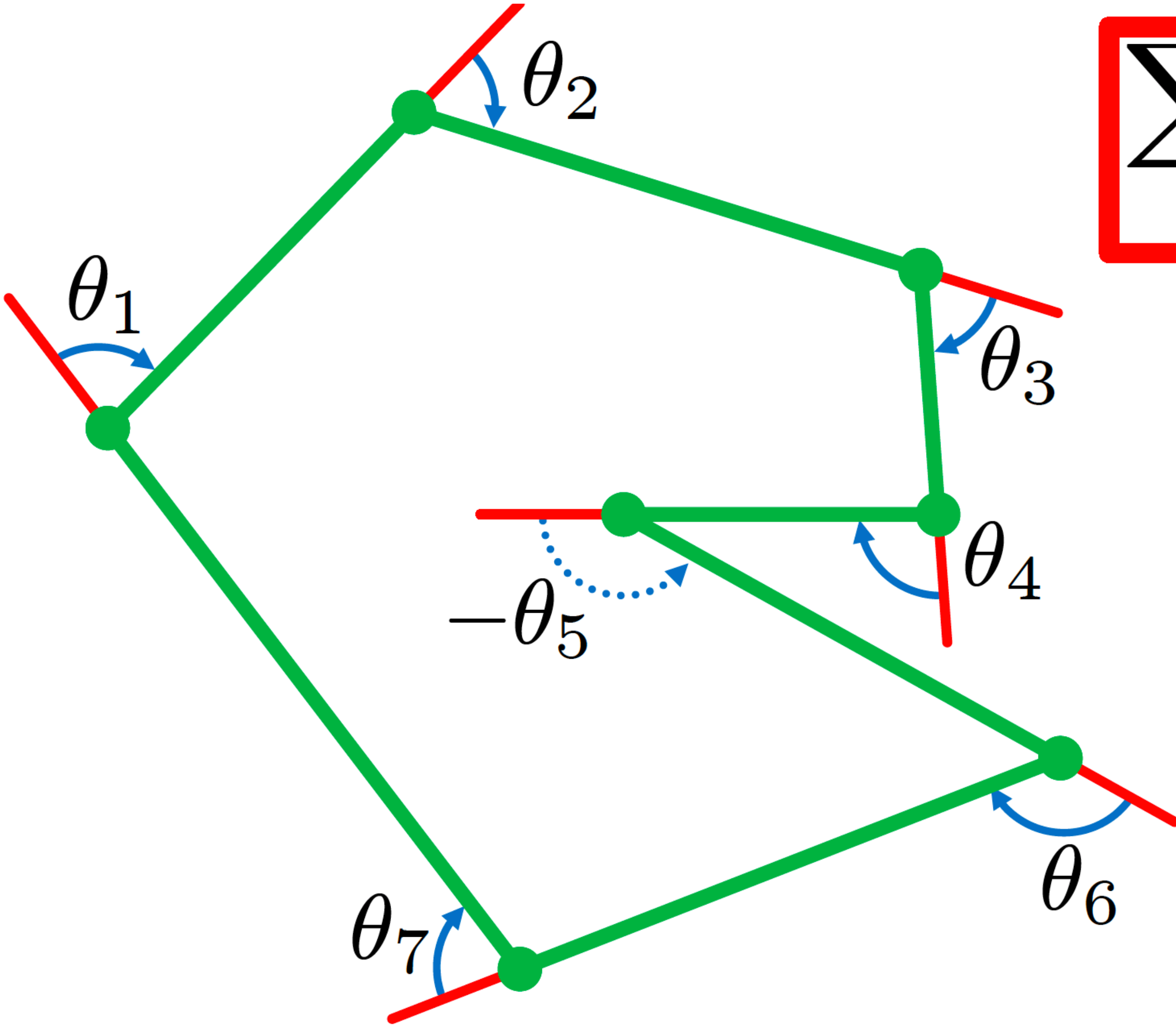
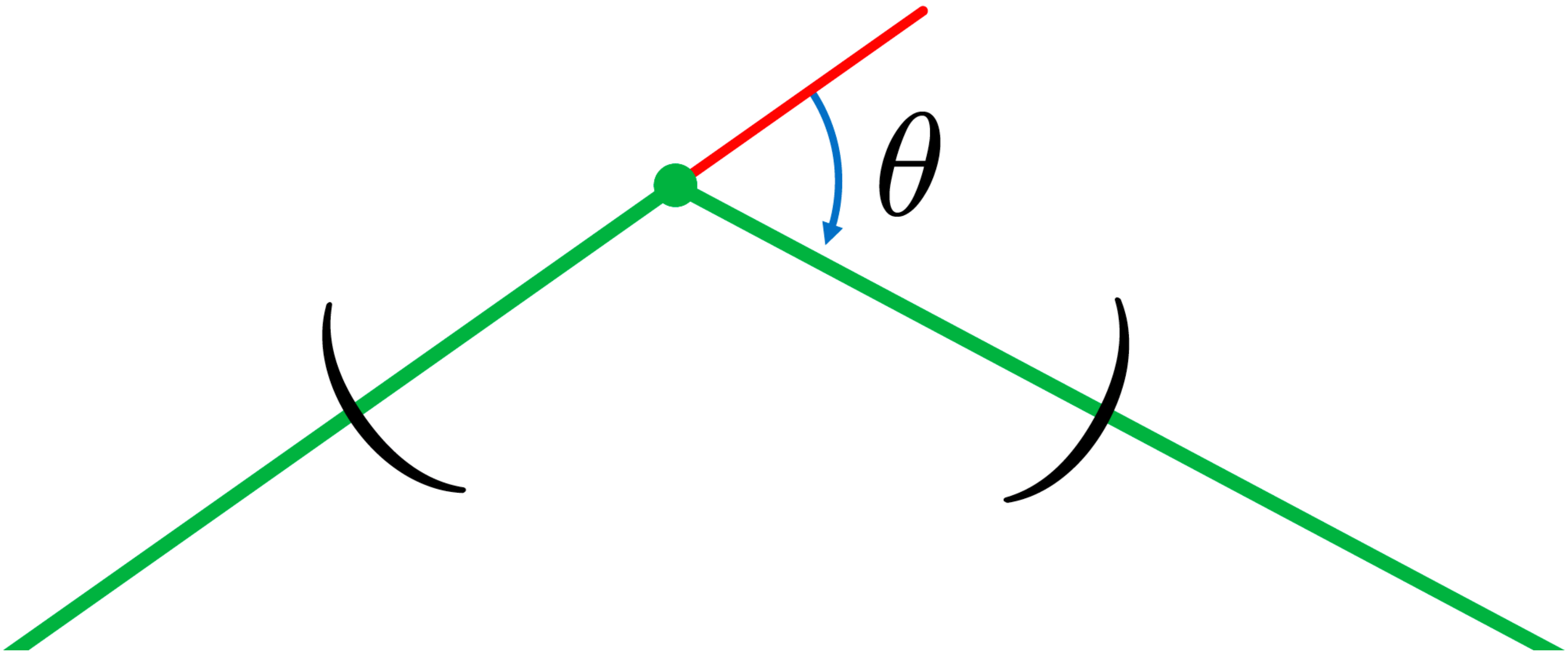
Key Observation



$$\boxed{\sum_i \theta_i = 2\pi k}$$

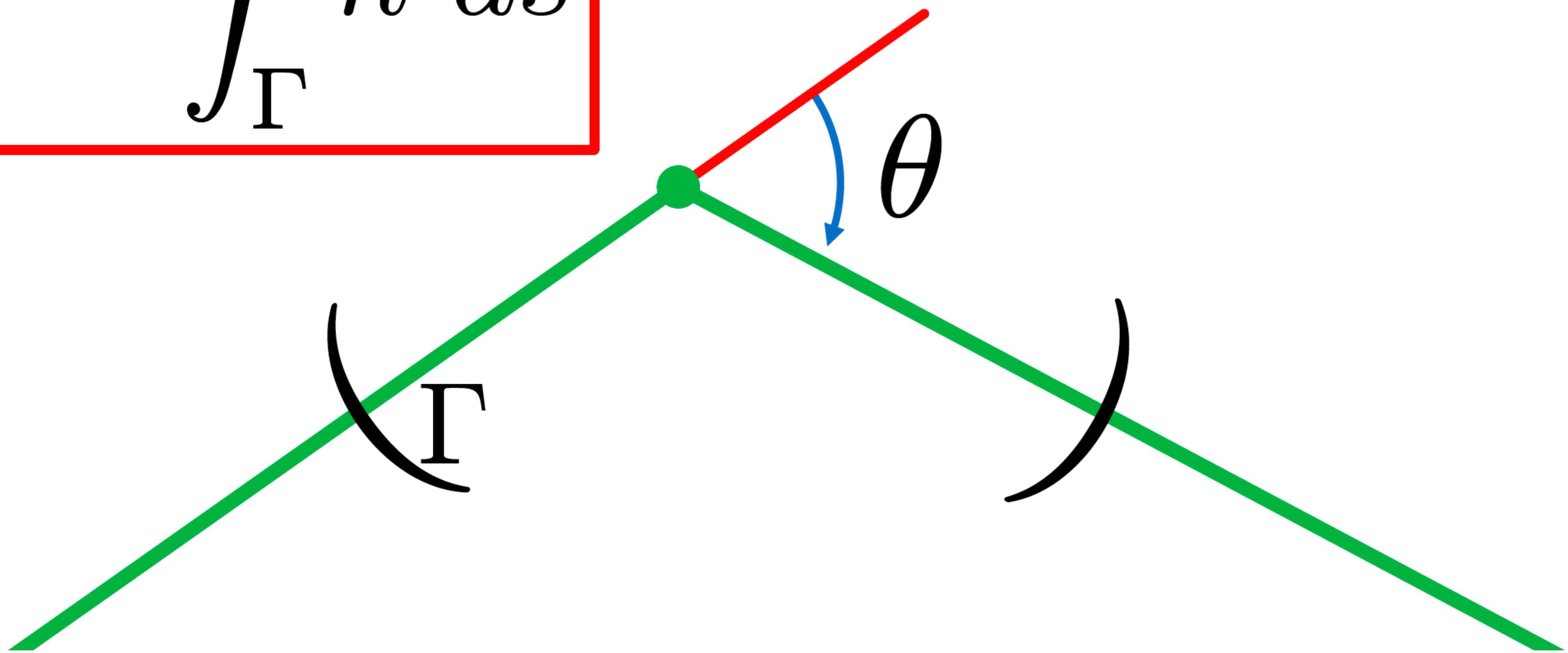
What's Going On?



Total change in curvature

What's Going On?

$$\boxed{\theta = \int_{\Gamma} \kappa \; ds}$$

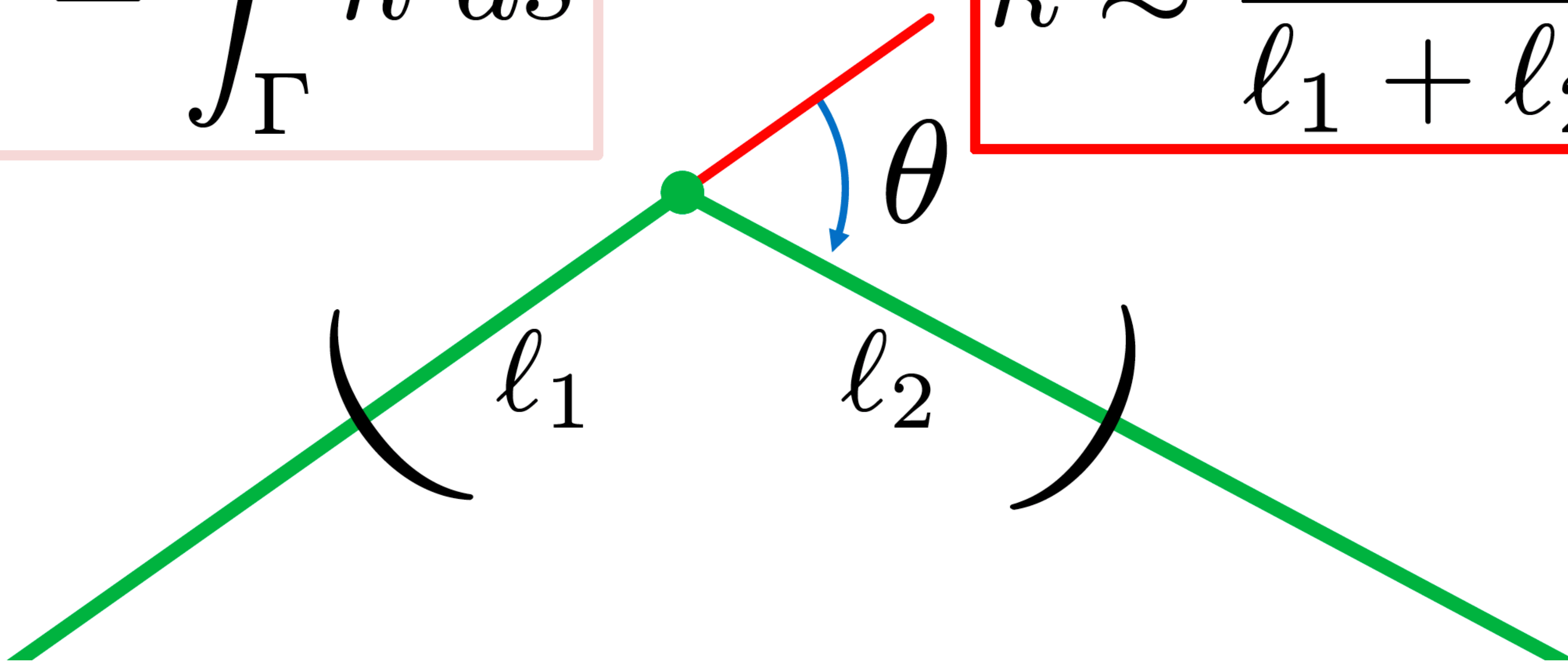


Total change in curvature

What's Going On?

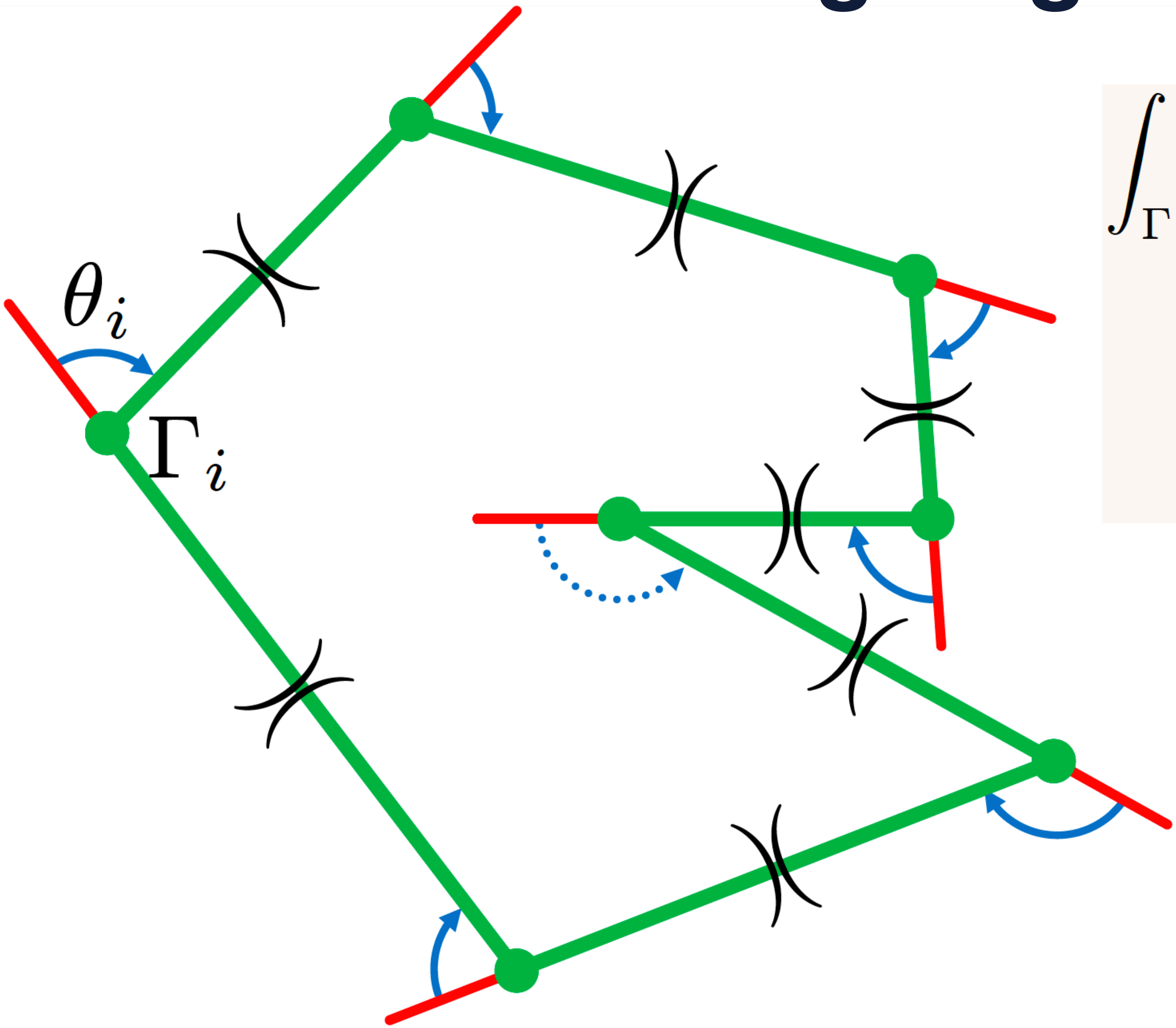
$$\theta = \int_\Gamma \kappa \, ds$$

$$\kappa \approx \frac{\theta}{\ell_1 + \ell_2}$$



Total change in curvature

Discrete Turning Angle Theorem



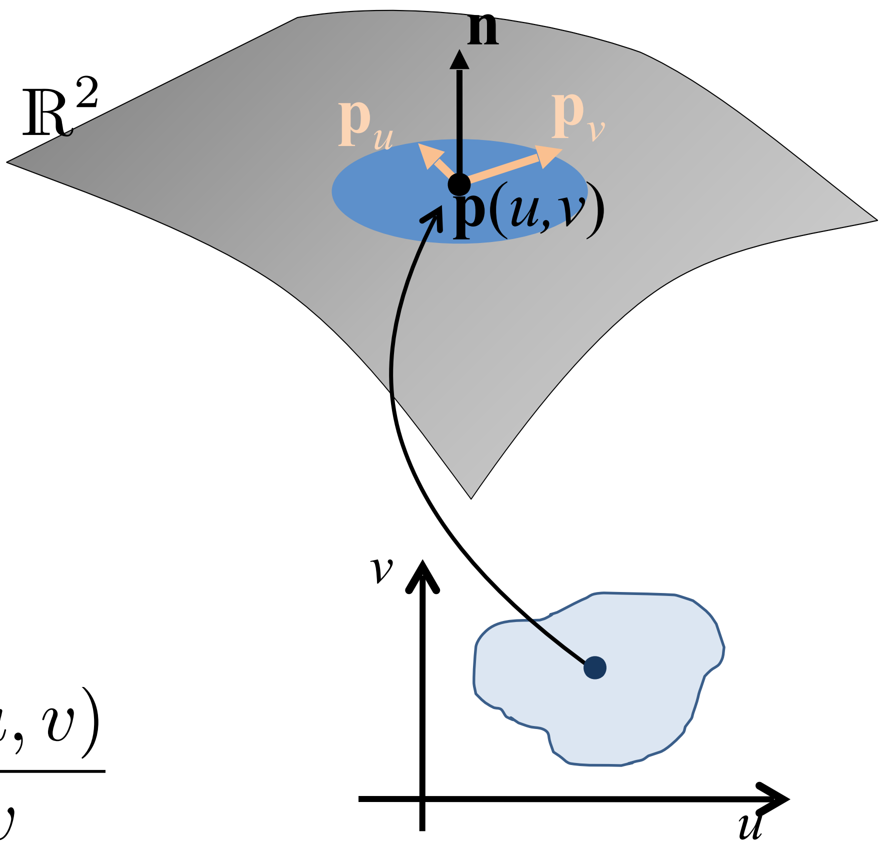
$$\int_\Gamma \kappa \; ds = \sum_i \int_{\Gamma_i} \kappa \; ds$$
$$= \sum_i \theta_i$$
$$= 2\pi k$$

GEOMETRY ON SURFACES

Surfaces, Parametric Form

- Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$



- Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

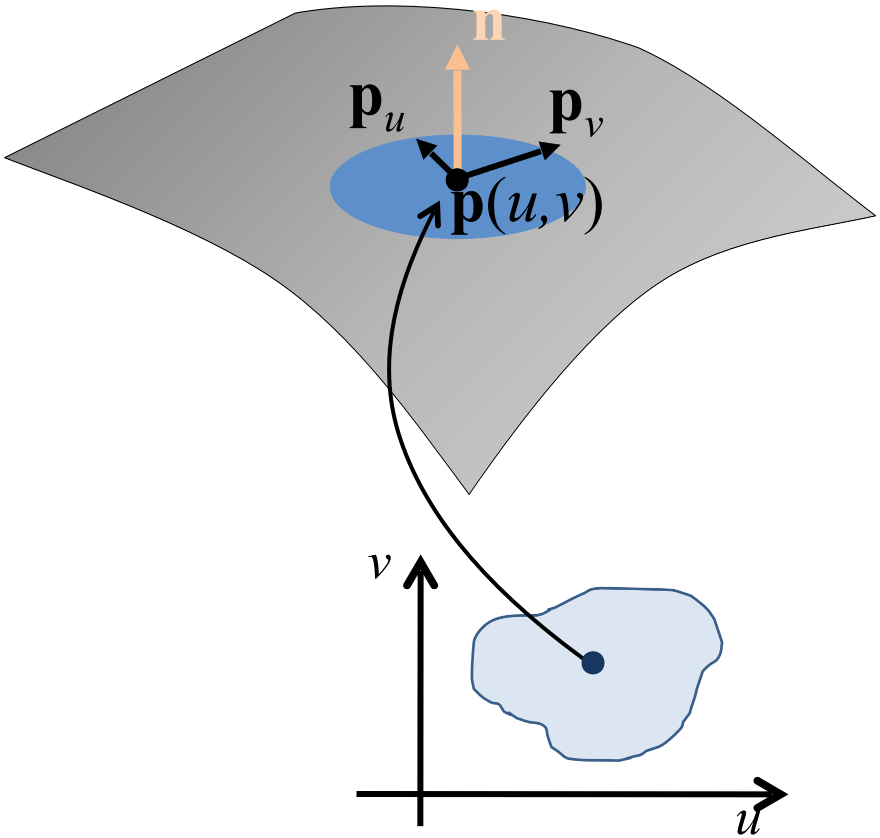
$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

These vectors don't have to be orthogonal

Surface Normals

- Surface normal:

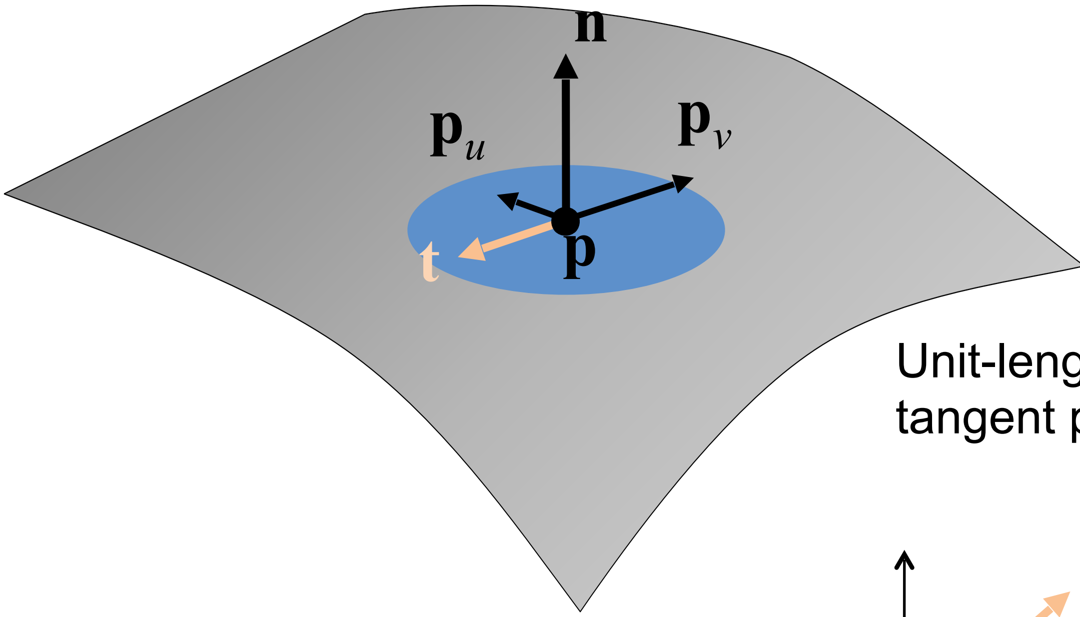
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$



- Assuming *regular* parameterization, i.e.,

$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$

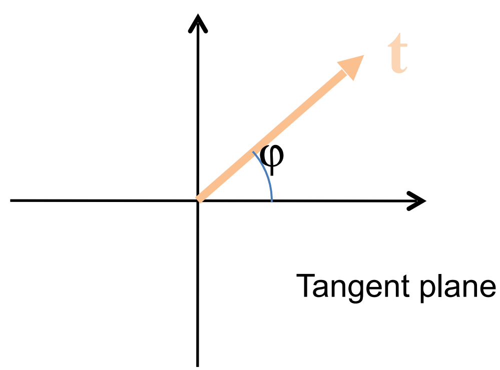
Normal Curvature



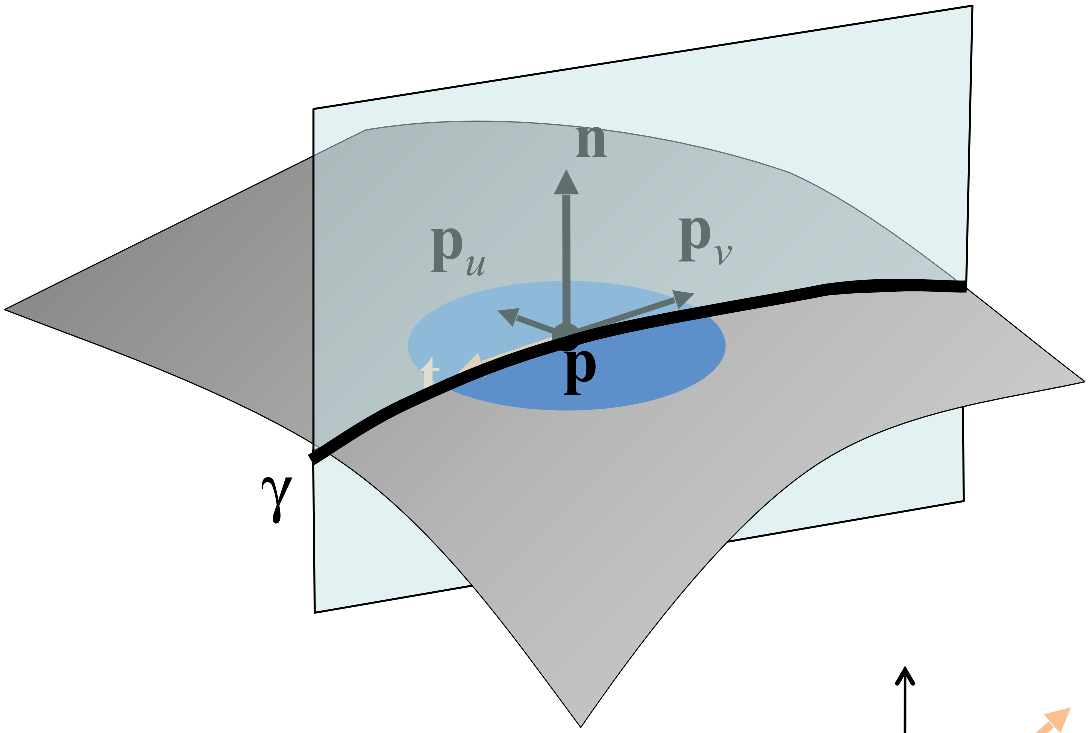
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

Unit-length direction **t** in the tangent plane (if $\mathbf{p}_u$ and $\mathbf{p}_v$ are orthogonal):

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\|\mathbf{p}_u\|} + \sin\varphi \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|}$$



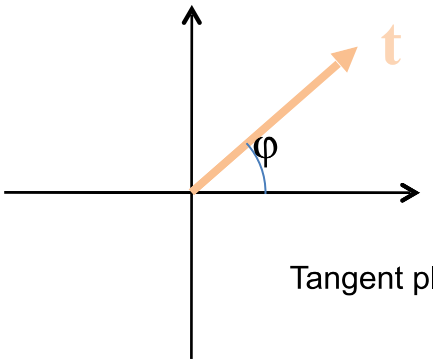
Normal Curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

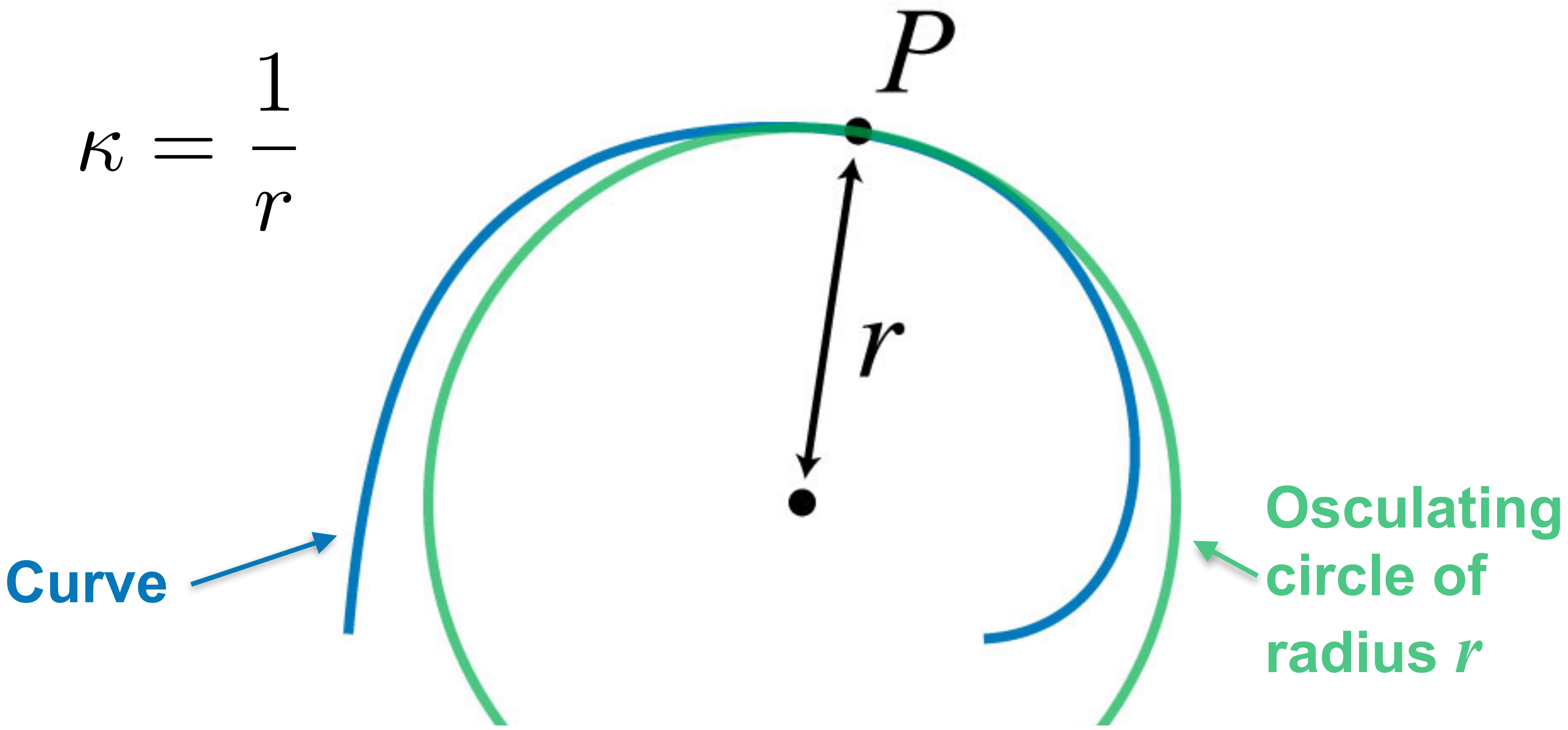
$$\kappa_n(\varphi) = \kappa(\gamma(\mathbf{p}))$$



Reminder: Radius of Curvature

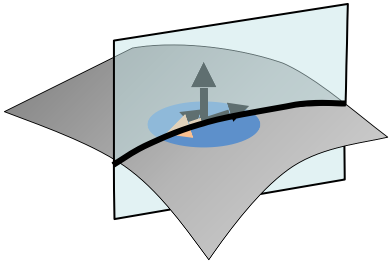
Curvature

$$\kappa = \frac{1}{r}$$



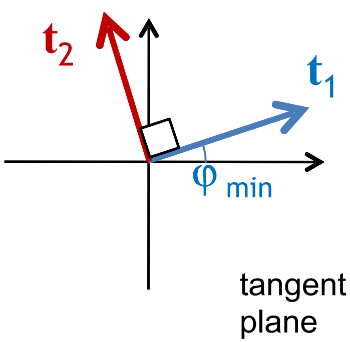
Surface Curvatures

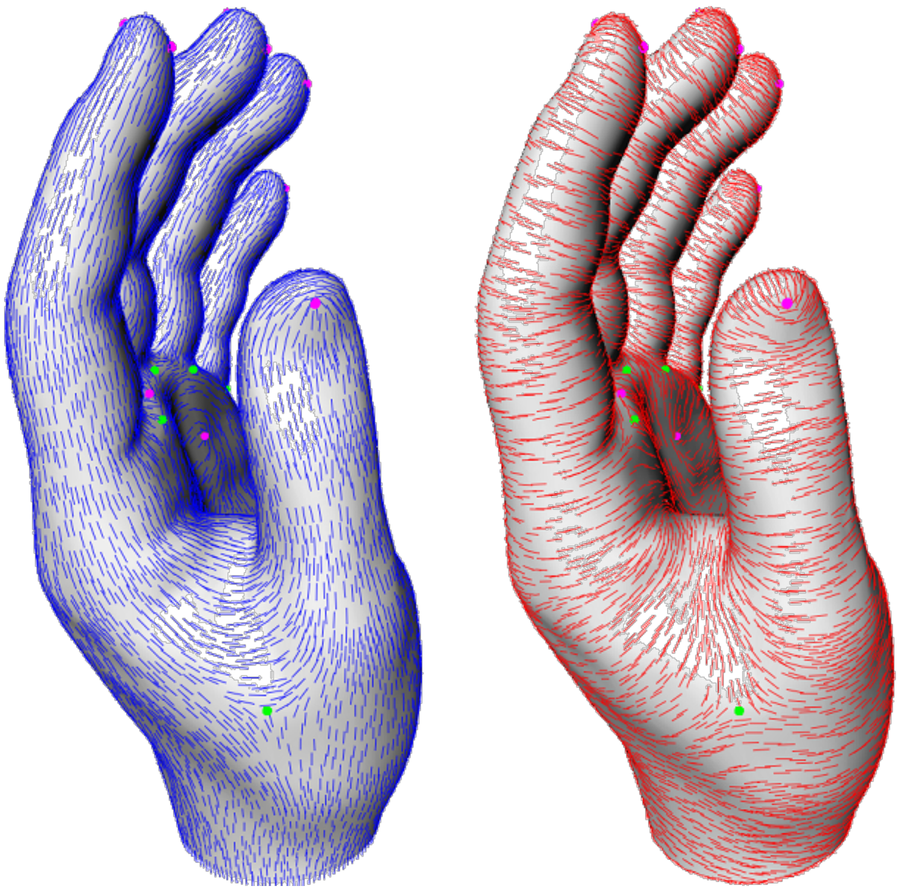
- Principal curvatures
  - Minimal curvature $\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$
  - Maximal curvature $\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$
- **Mean curvature** $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\varphi)d\varphi$
- **Gaussian curvature** $K = \kappa_1 \cdot \kappa_2$



Principal Directions

- Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}

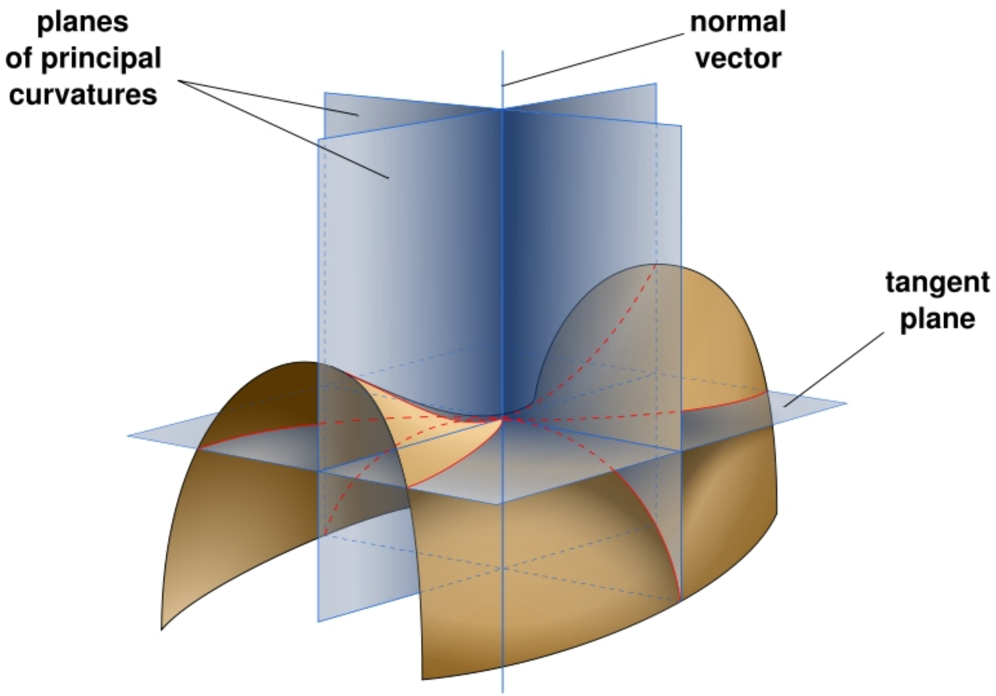




min curvature

max curvature

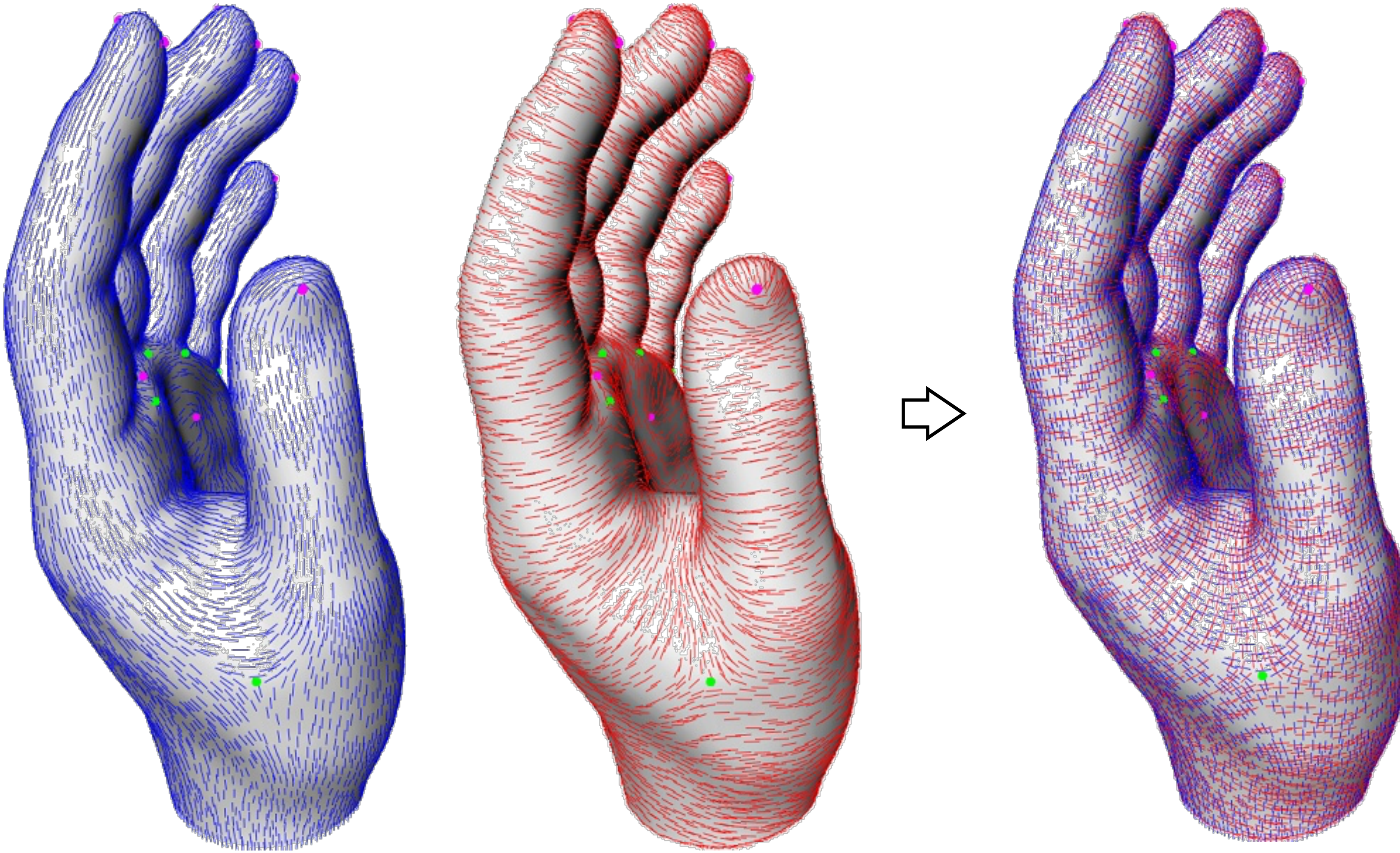
Principal Directions



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

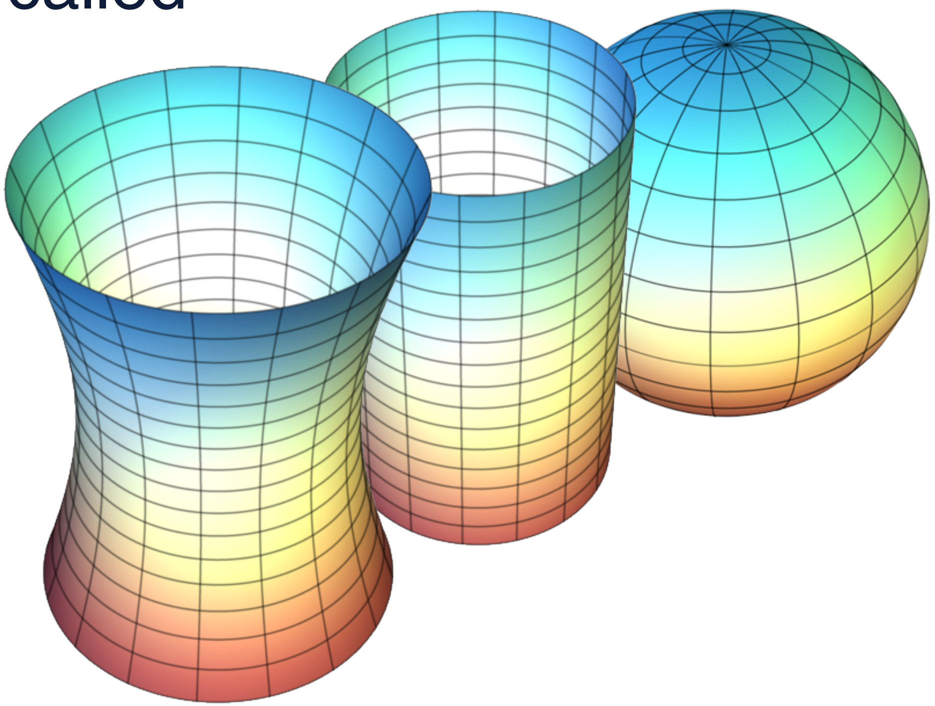
$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \quad \varphi = \text{angle with } \mathbf{t}_1$$

Principal Directions



Classification

- A point $\mathbf{p}$ on the surface is called
  - Elliptic, if $K > 0$
  - Parabolic, if $K = 0$
  - Hyperbolic, if $K < 0$

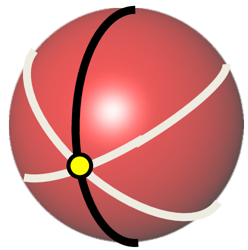


- Developable surface
  iff $K = 0$
  - can be mapped to the plane without distortion

Local Surface Shape By Curvatures

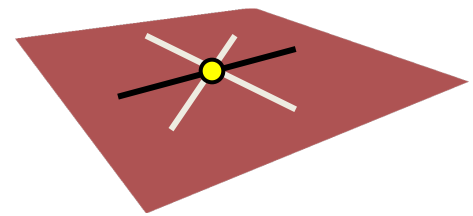
Isotropic:
all directions are principal directions

$K > 0, \kappa_1 = \kappa_2$



spherical (umbilical)

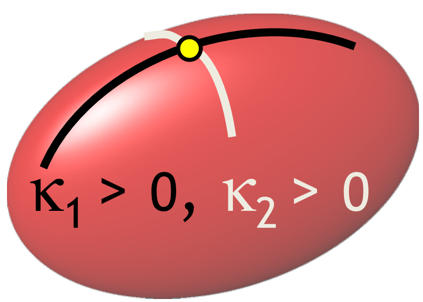
$K = 0$



planar

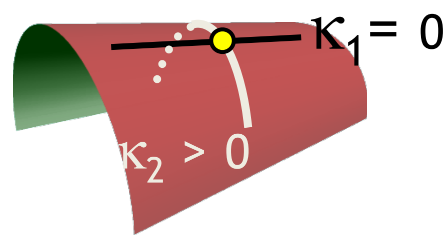
Anisotropic:
2 distinct principal directions

$K > 0$



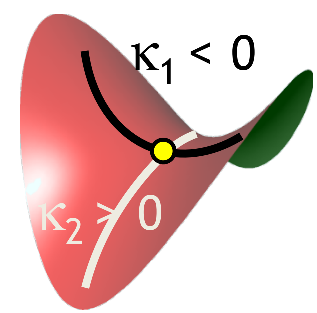
elliptic

$K = 0$



parabolic

$K < 0$



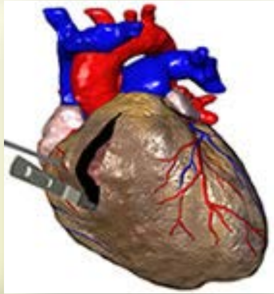
hyperbolic

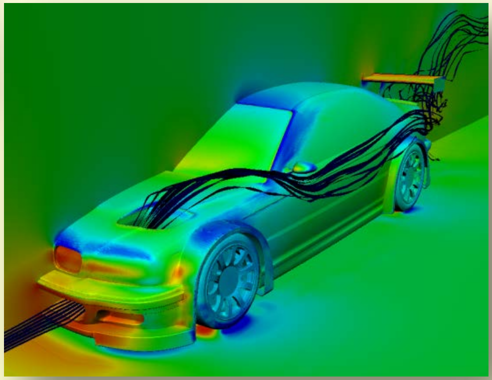
INTRODUCTION TO GEOMETRY PROCESSING

UC San Diego

Application Areas

- Computer games
- Movie production
- Engineering
- Medical applications
- Architecture
- etc.





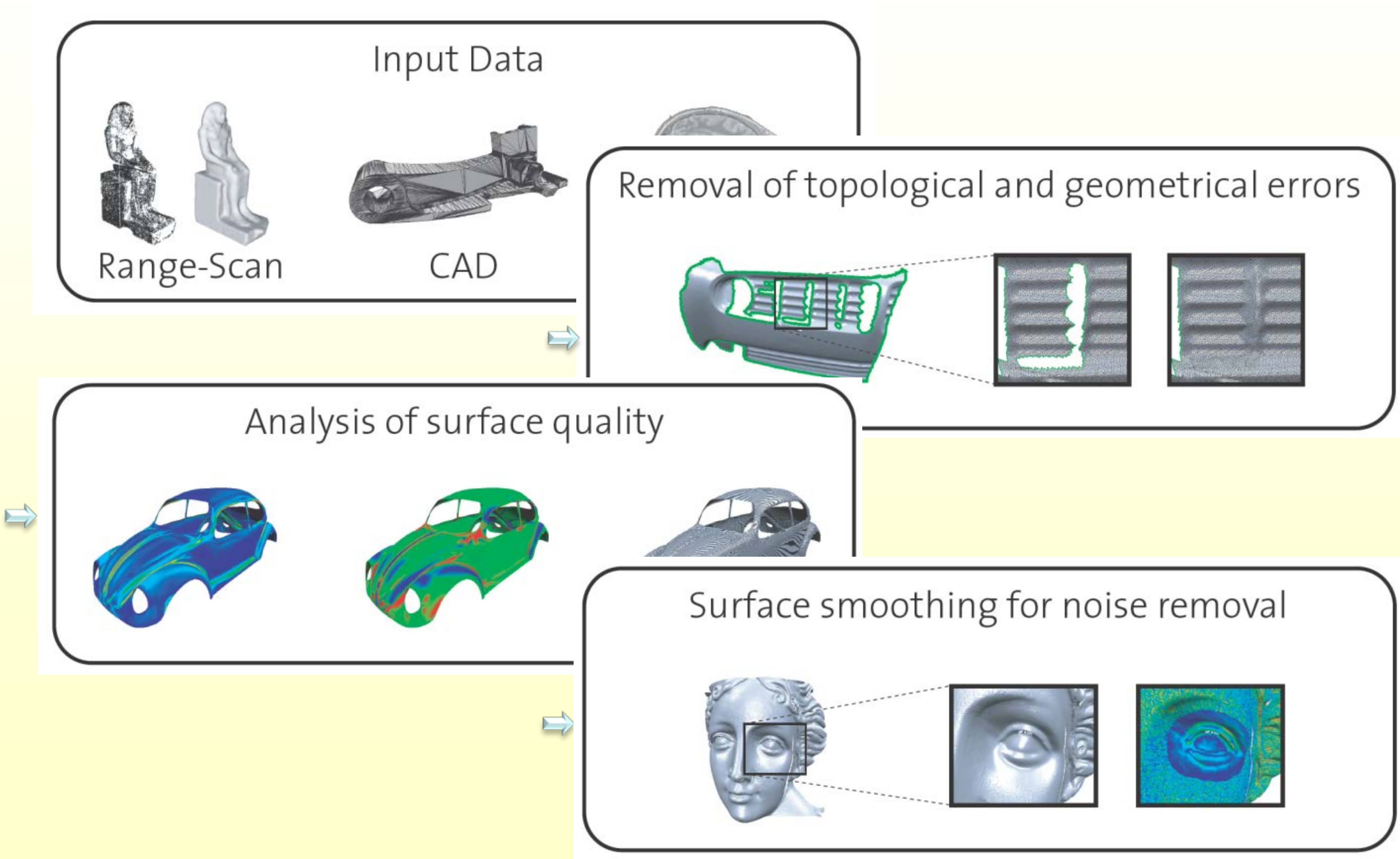
What is Geometry Processing About?

- Acquiring
- Analyzing/Improving
- Manipulating

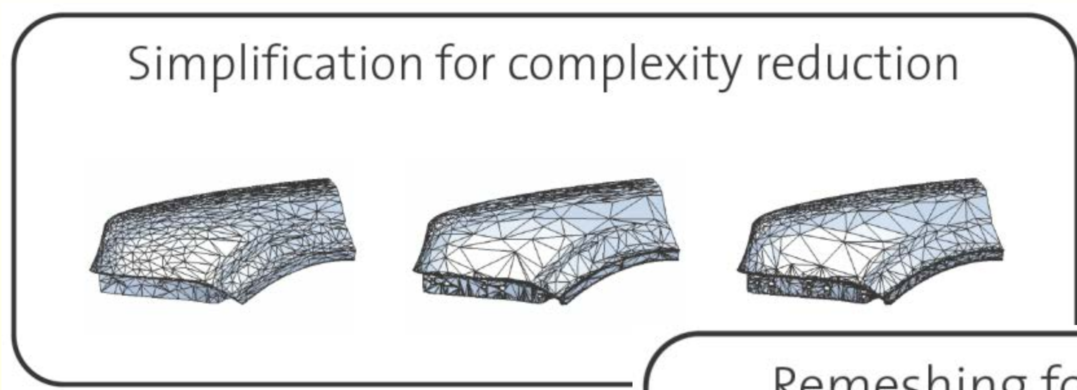
3D Models

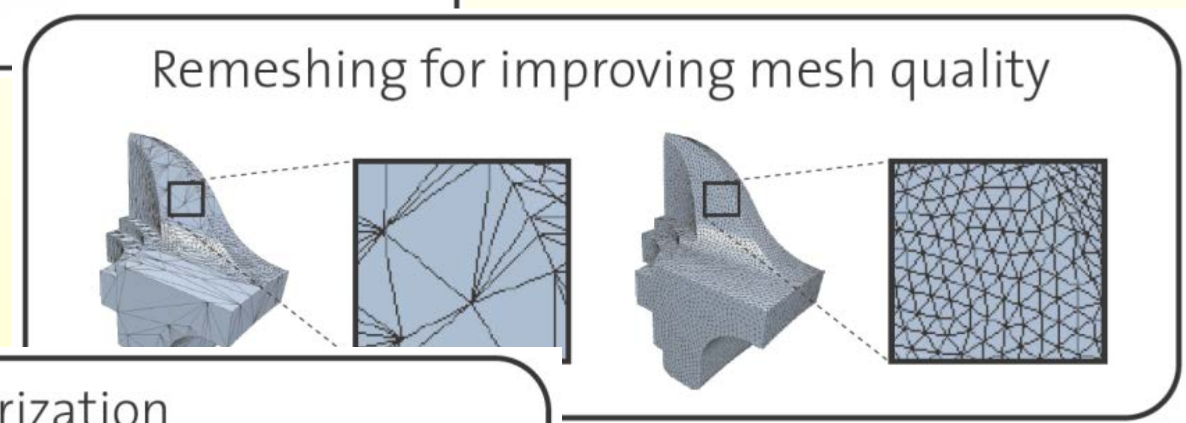
A Geometry Processing Pipeline

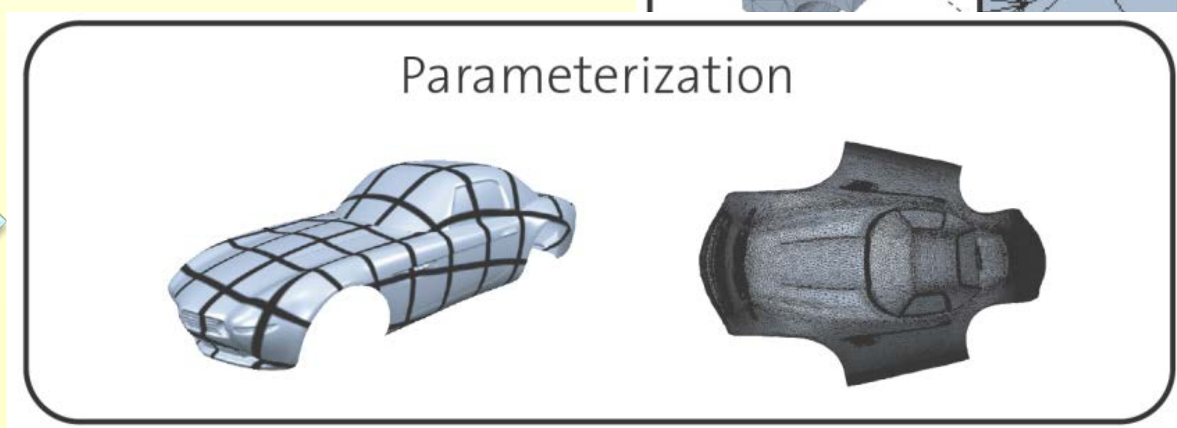
Low Level Algorithms



A Geometry Processing Pipeline



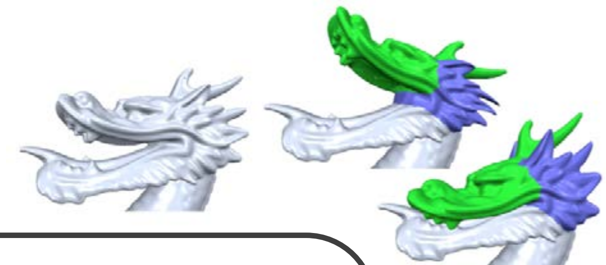




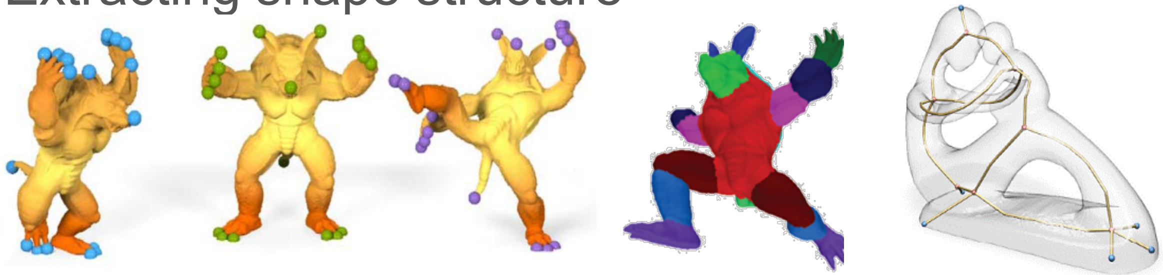
A Geometry Processing Pipeline
High Level Algorithms

Freeform and multiresolution modeling

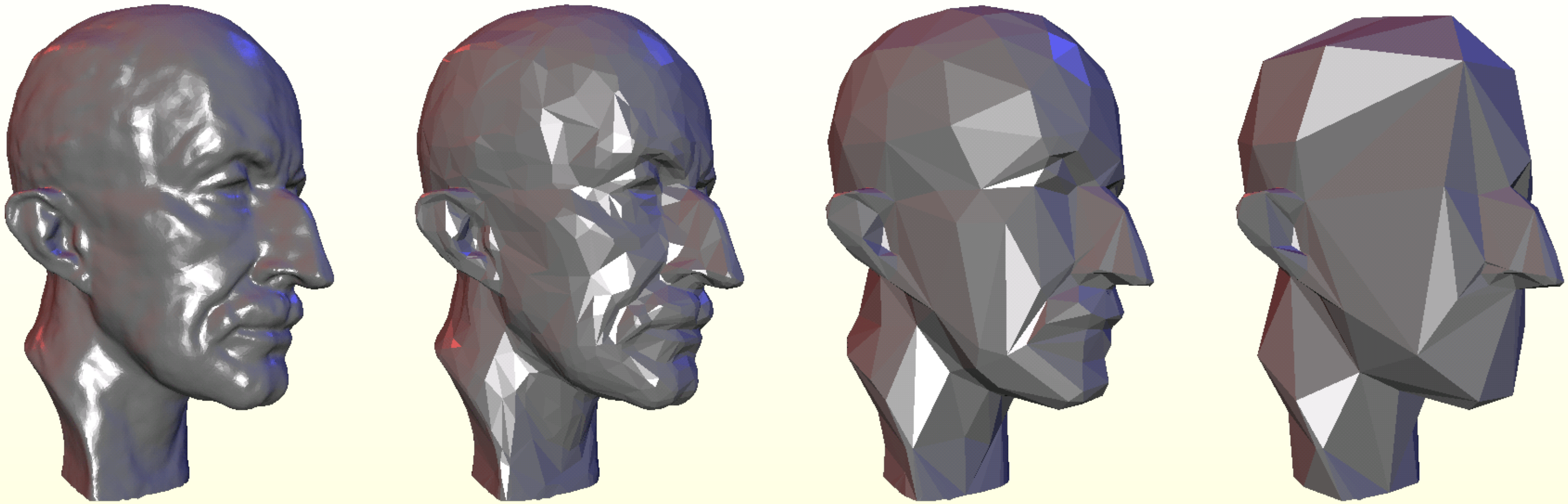
Deformation and editing

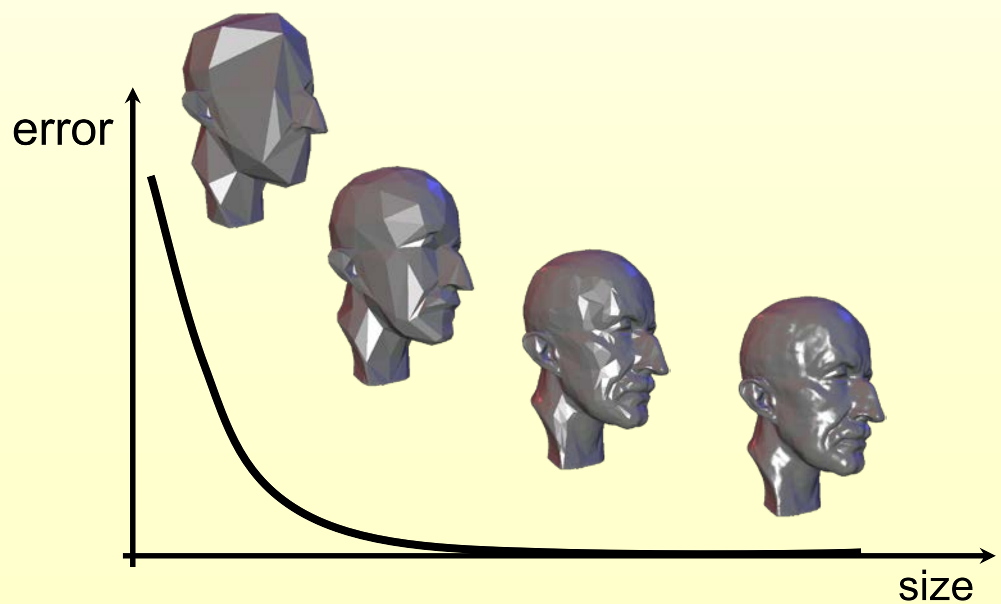


Extracting shape structure



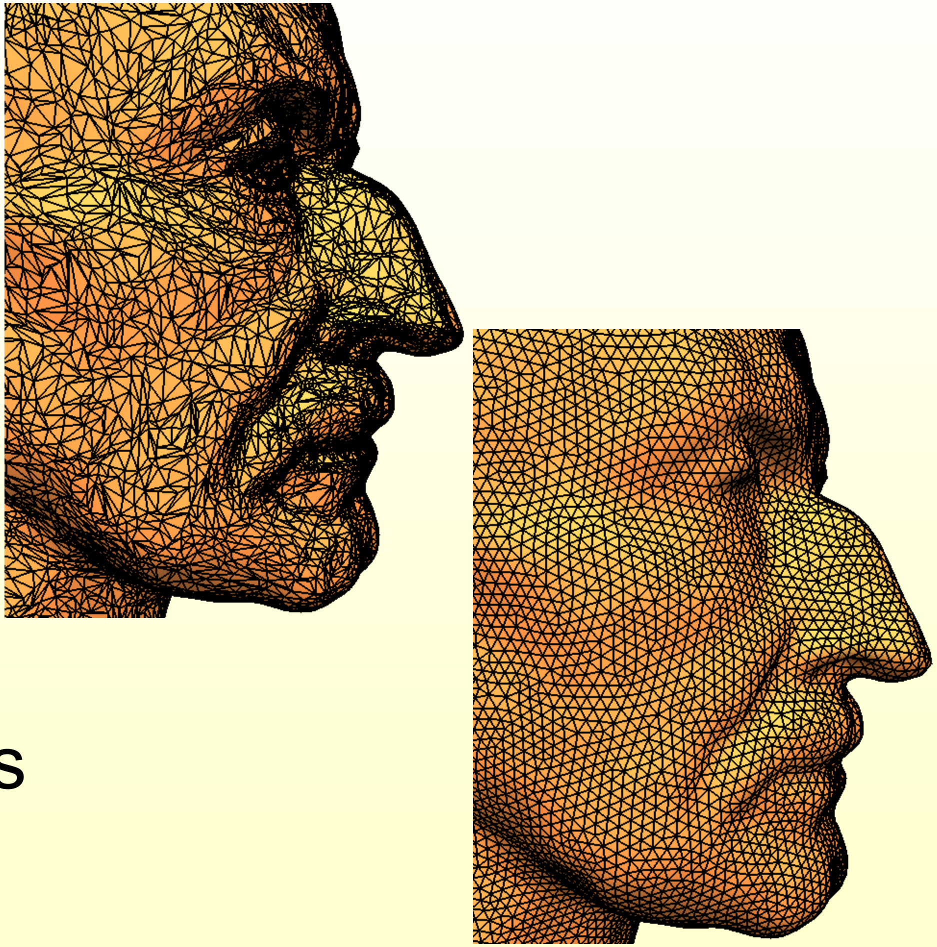
Simplification



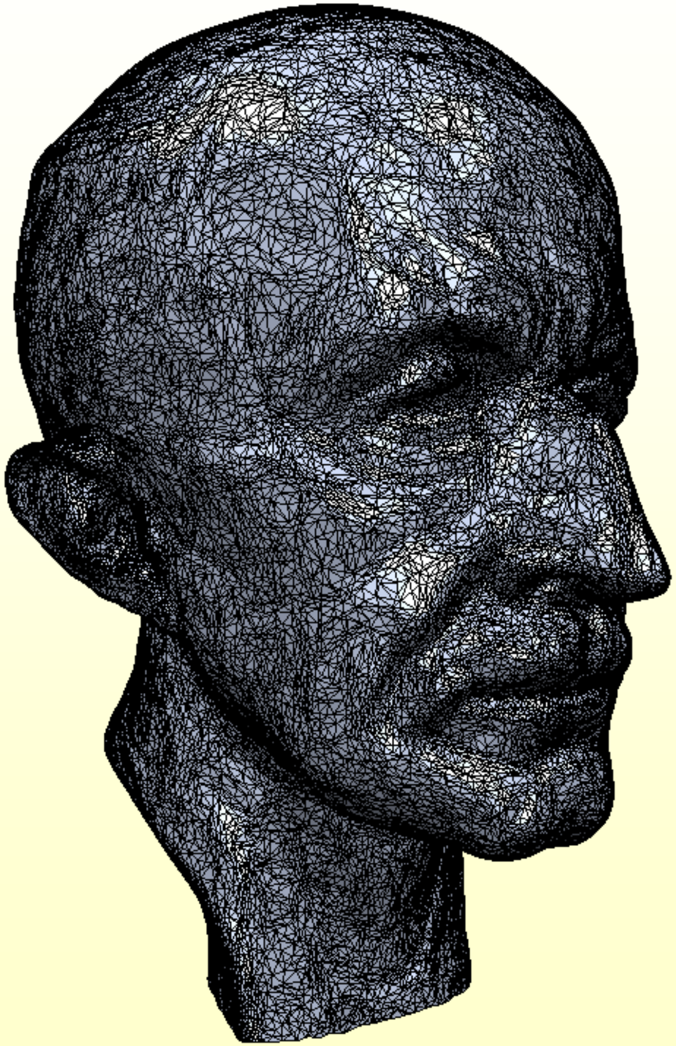


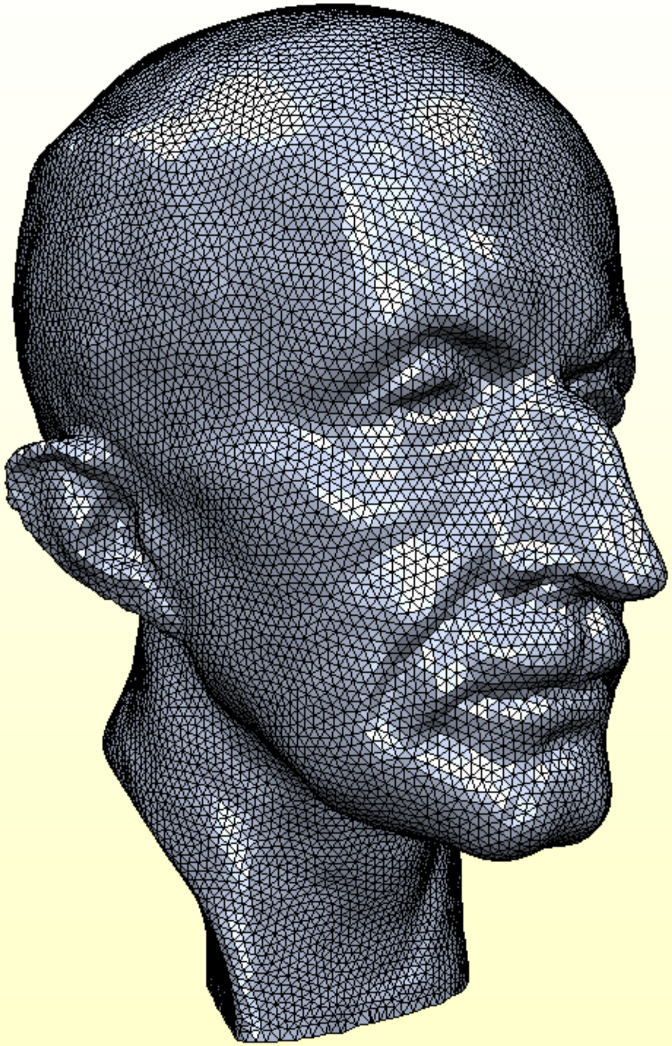
Mesh Quality Criteria

- Smoothness
  - Low geometric noise
- Adaptive tessellation
  - Low complexity
- Triangle shape
  - Numerical robustness



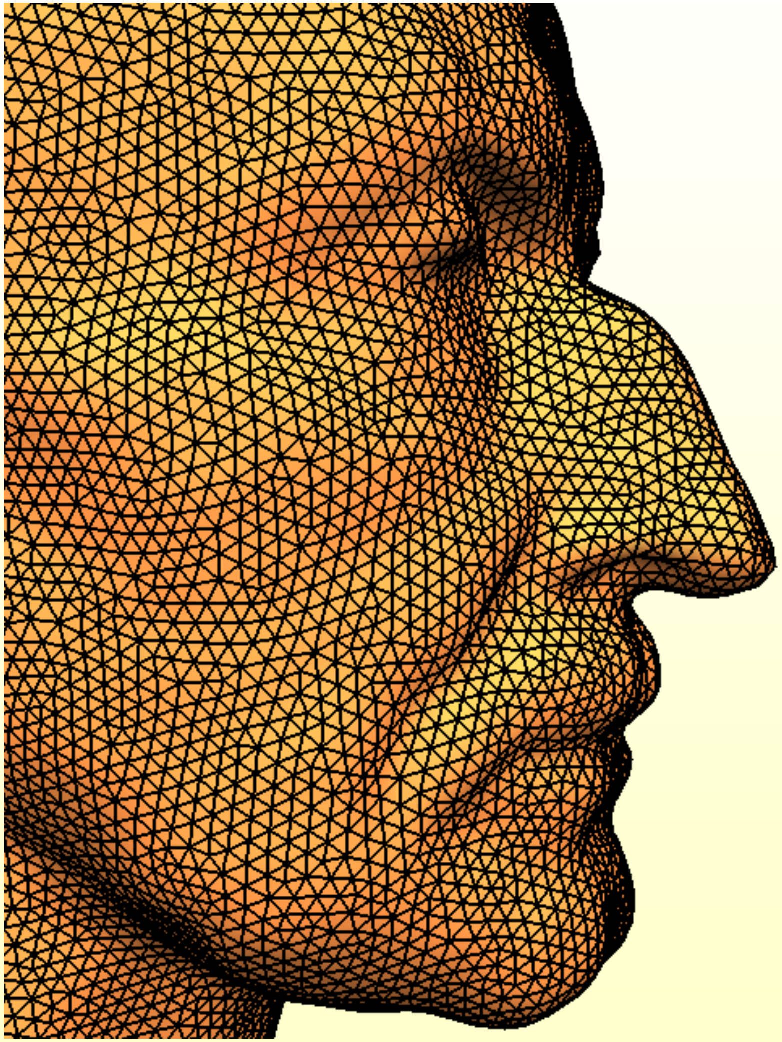
What is a Good Mesh?





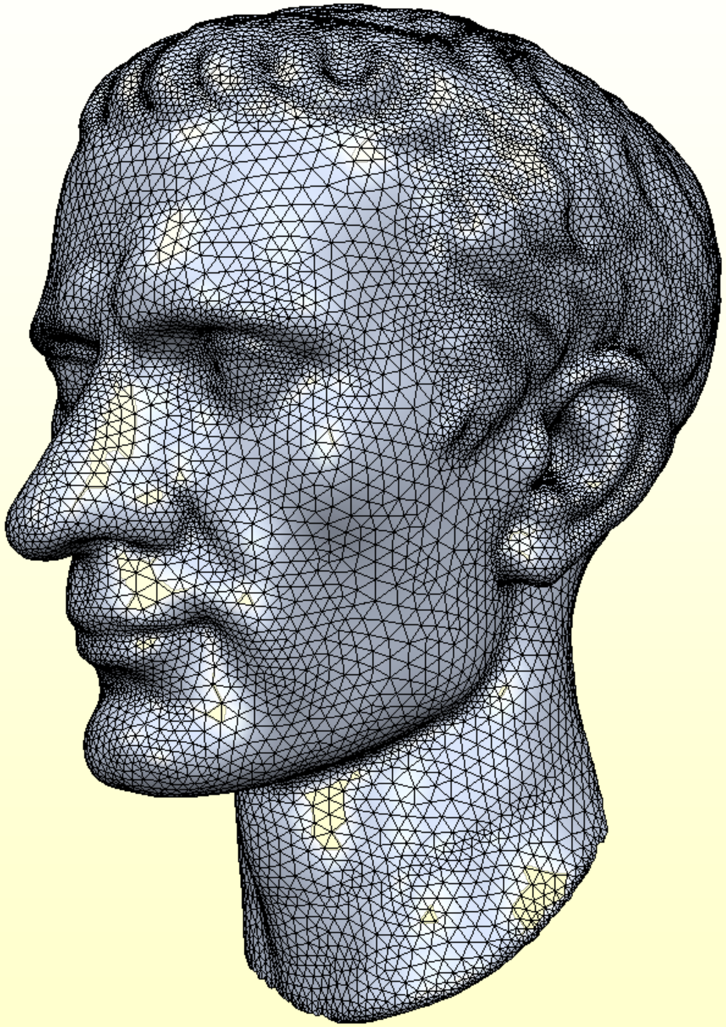
What is a Good Mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6



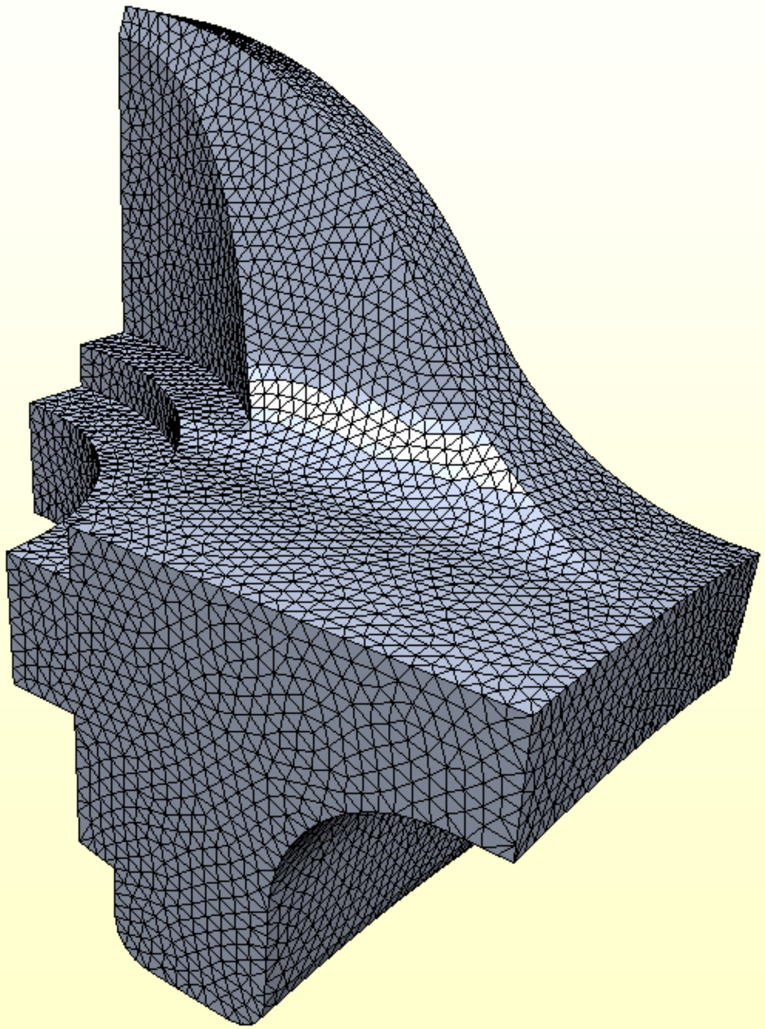
What is a Good Mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling



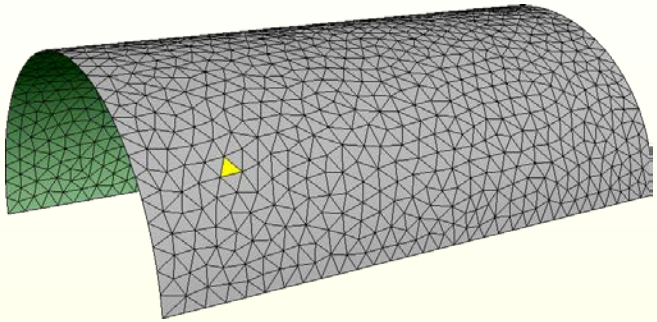
What is a Good Mesh?

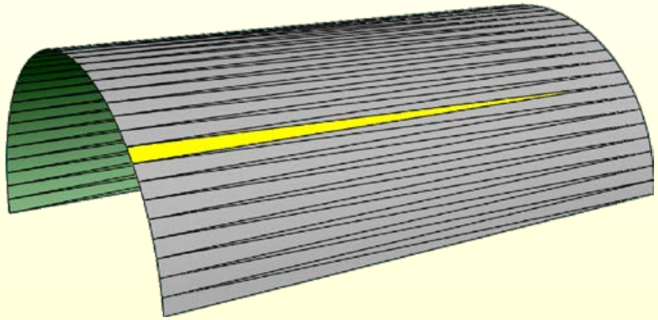
- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling
- Feature preservation

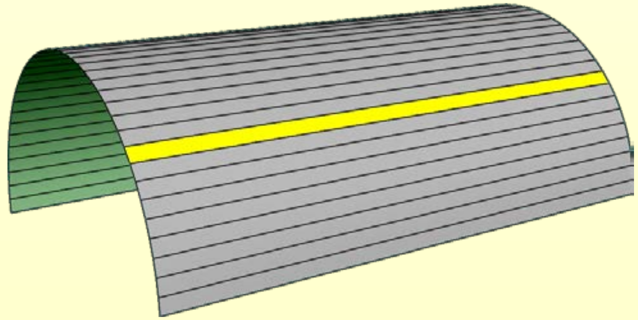


What is a Good Mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling
- Feature preservation
- Alignment to curvature lines
- Isotropic vs. anisotropic

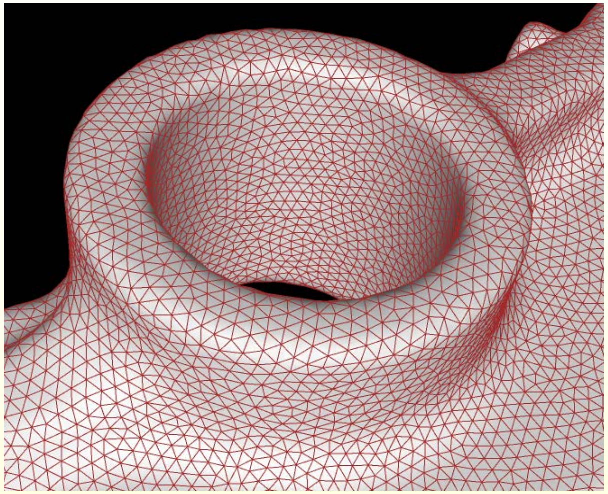


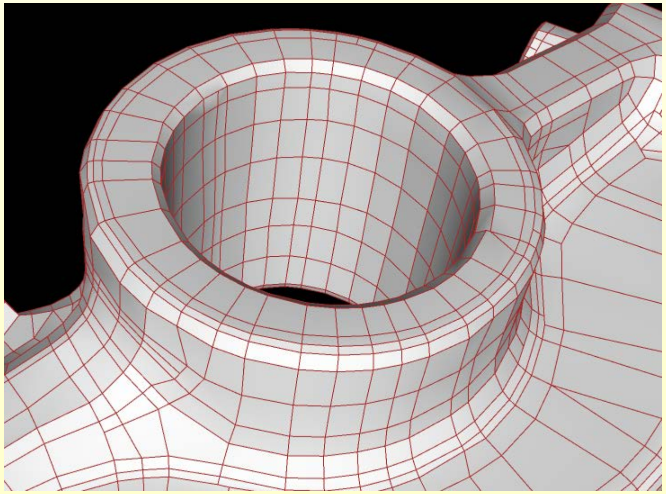




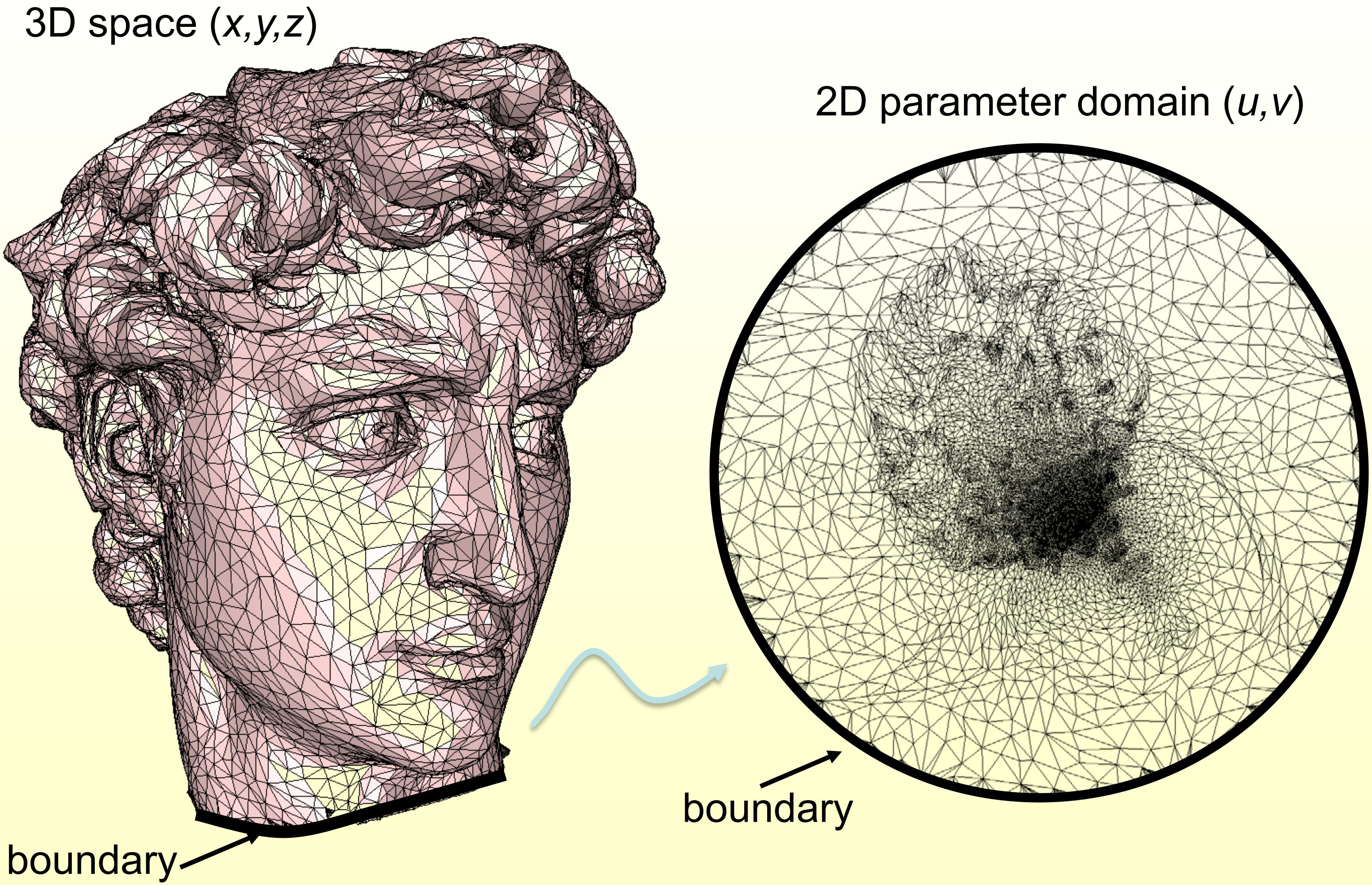
What is a Good Mesh?

- Equal edge lengths
- Equilateral triangles
- Valence close to 6
- Uniform vs. adaptive sampling
- Feature preservation
- Alignment to curvature lines
- Isotropic vs. anisotropic
- Triangles vs. quadrangles



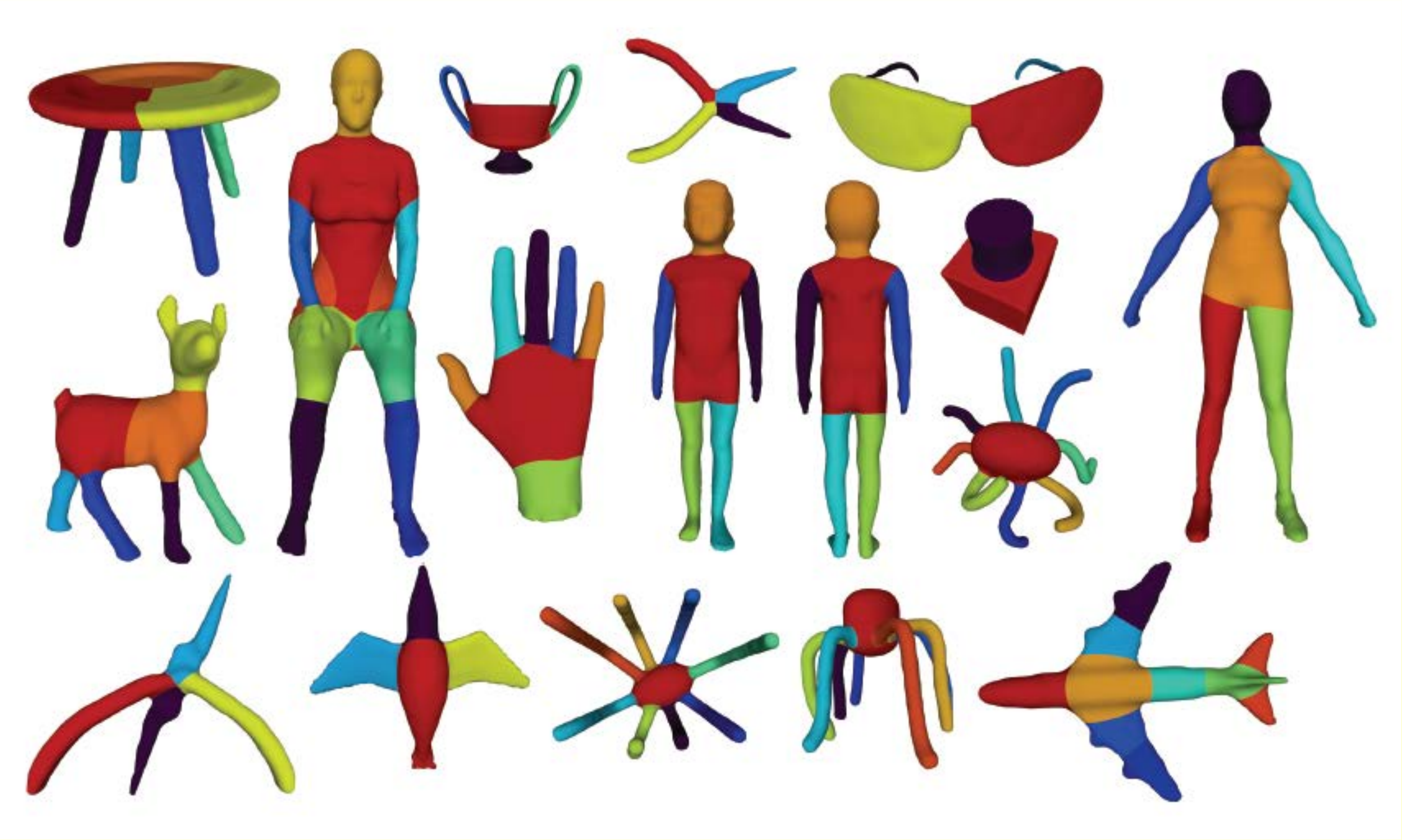


Parametrization

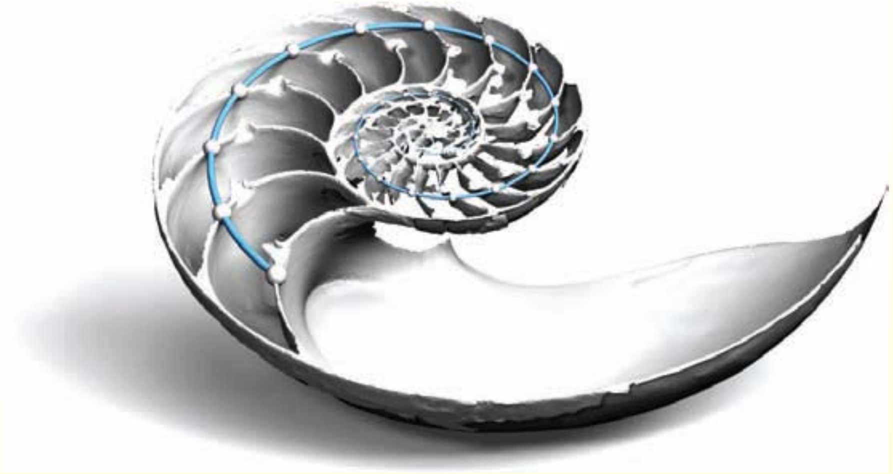


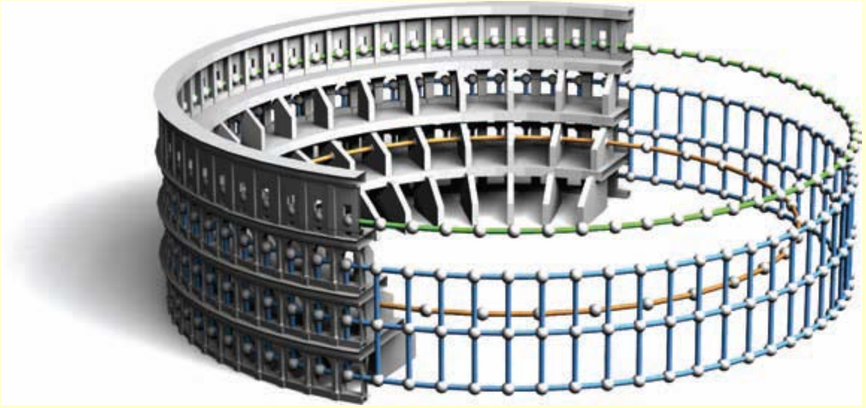
Application -- Texture Mapping

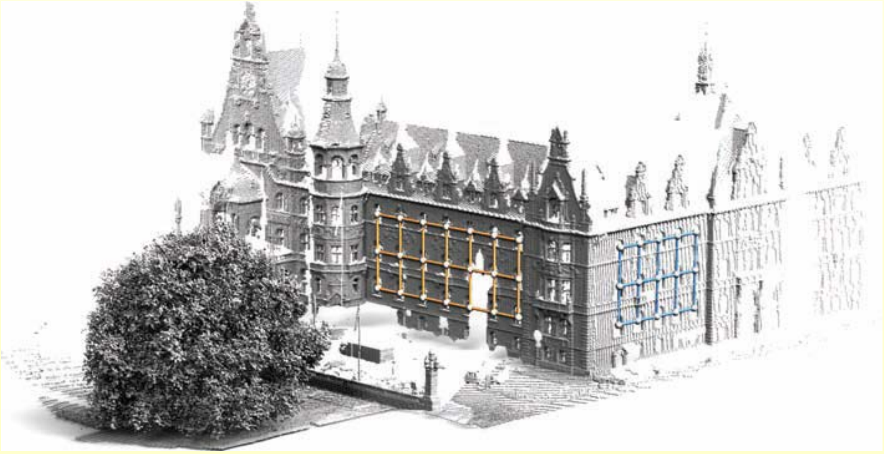
Segmentation



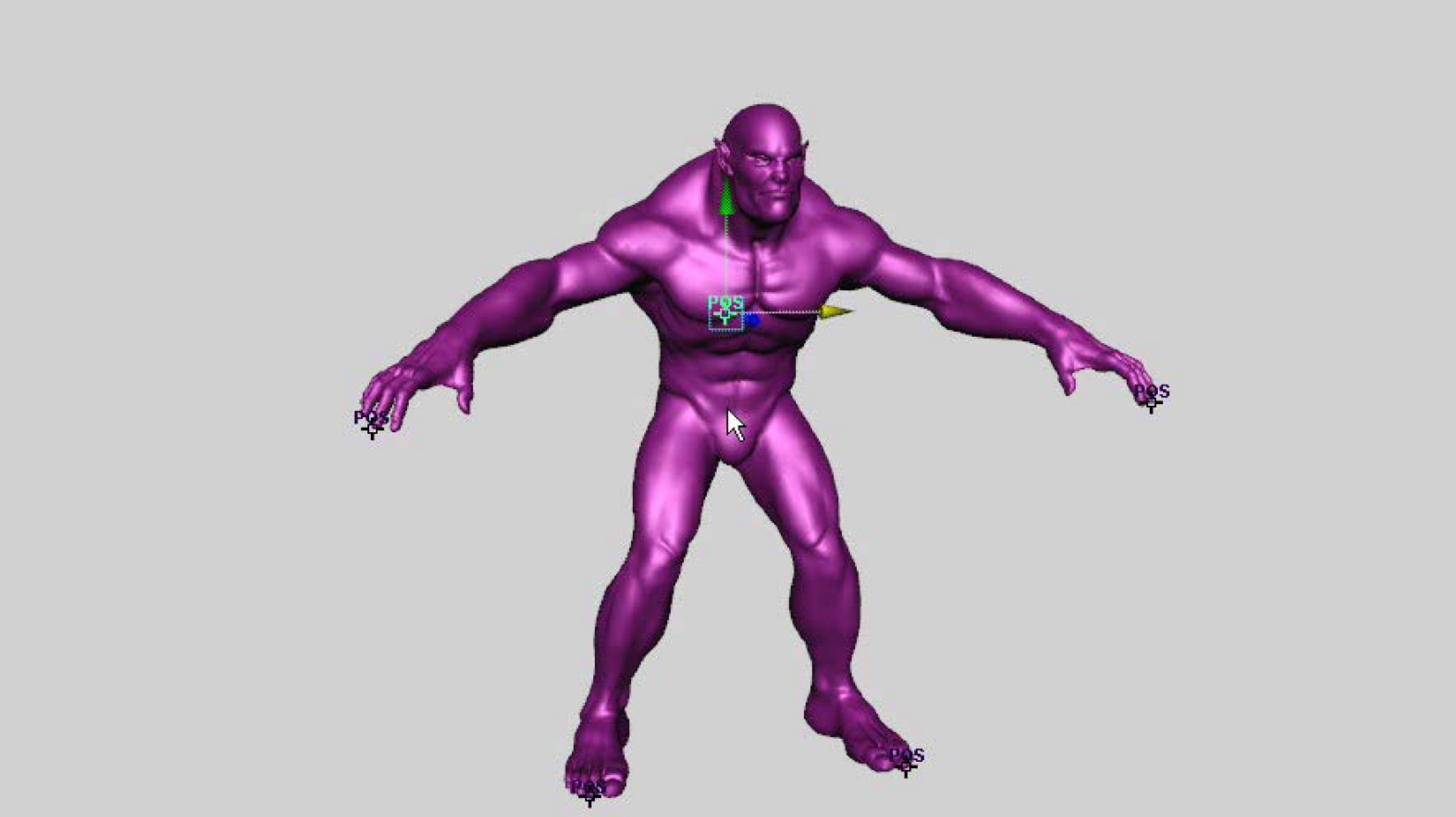
Symmetry Detection







Deformation / Manipulation



Next Lecture

- 3D Deep Learning on Point Clouds