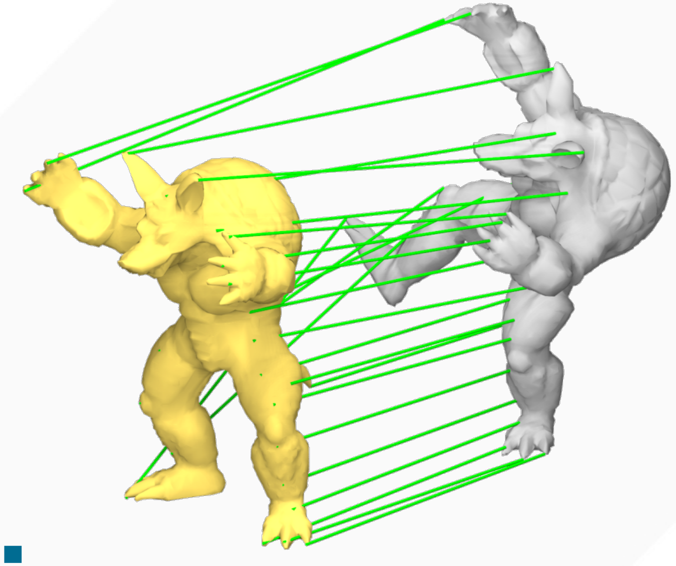

Lecture 16:

Intrinsic Geometry

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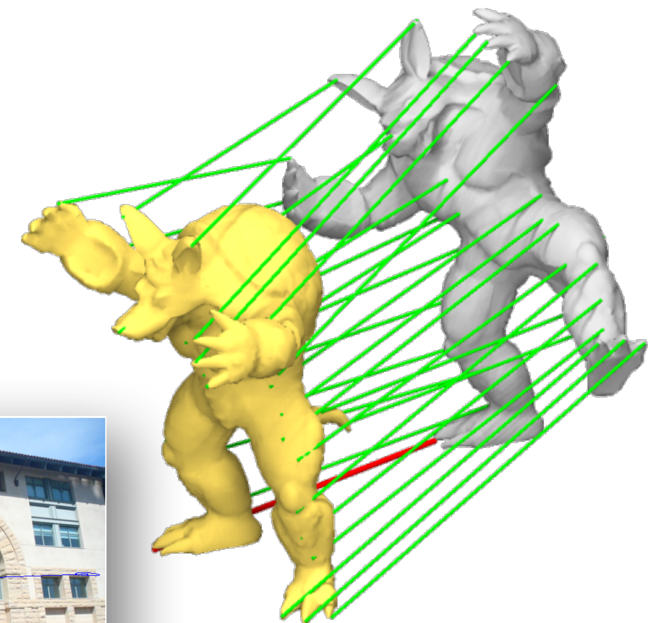
Mar 6, 2018

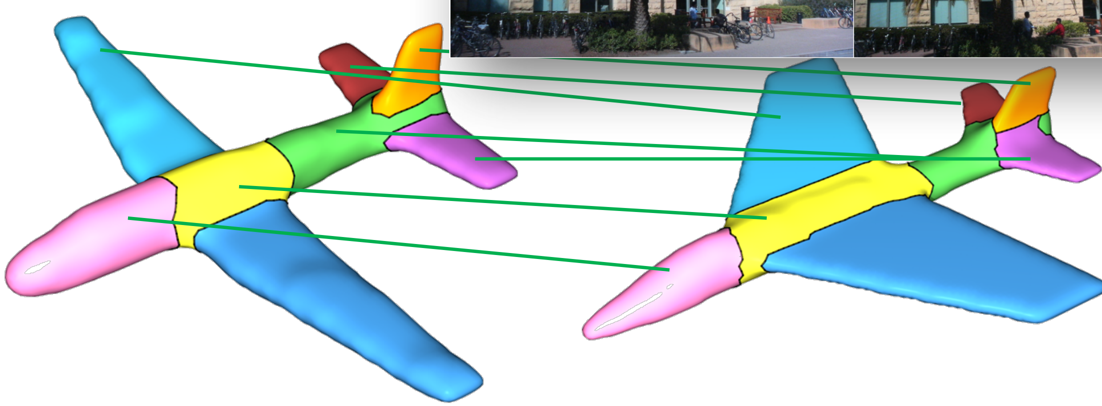
Slides ack: Leo Guibas, Yaron Lipman, Peter Huang, Vova Kim, Maks Ovsjanikov, Michael Bronstein

Alignment and Registration of Data Sets

Mapping Between Data Sets

- Multiscale mappings
  - Point/pixel level
  - part level



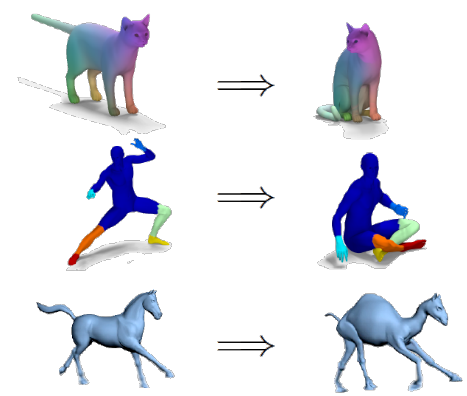


Maps capture what is the same or similar across two data sets

Why Do We Care About Maps and Alignments?

- To stitch data together

- To transfer information



- To compute distances and similarities

- To perform joint analysis

Extrinsic vs. Intrinsic Alignment

- Coordinate root mean squared distance

$$\mathrm{cRMS}^2(\mathbf{P},\mathbf{Q}) = \min_{\mathbf{R},\mathbf{t}} \frac{1}{n} \sum_{i=1}^{n} ||\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i||^2$$

estimate transform

- Distance root mean squared distance

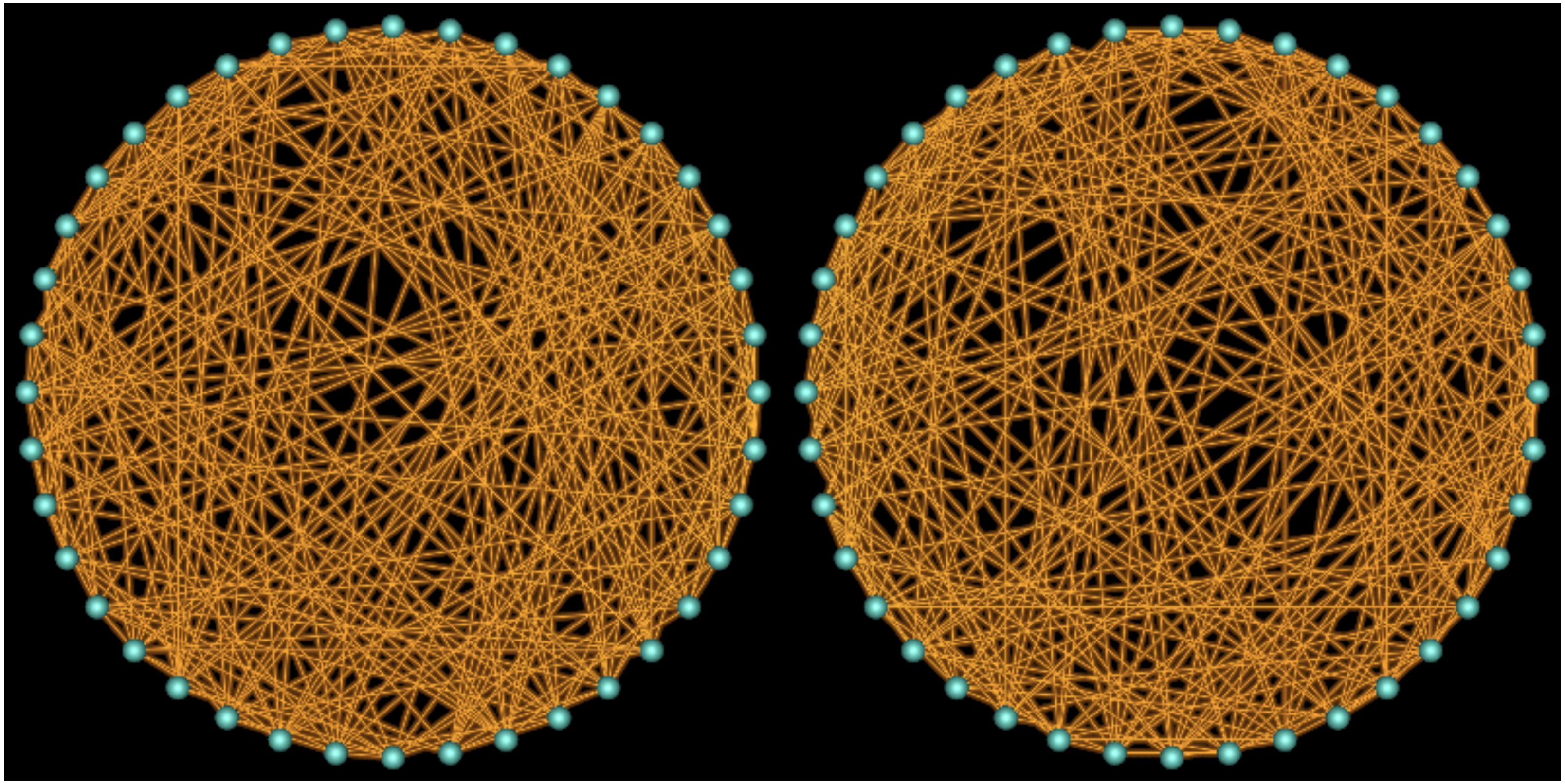
$$\mathrm{dRMS}^2(\mathbf{P},\mathbf{Q}) = \frac{1}{n^2} \min_{\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} (||\mathbf{p}_i - \mathbf{p}_j|| - ||\mathbf{q}_{\sigma(i)} - \mathbf{q}_{\sigma(j)}||)^2$$

estimate correspondences

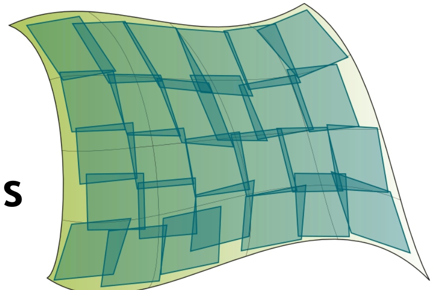
metric space, intrinsic alignment

Gromov-Hausdorff distances

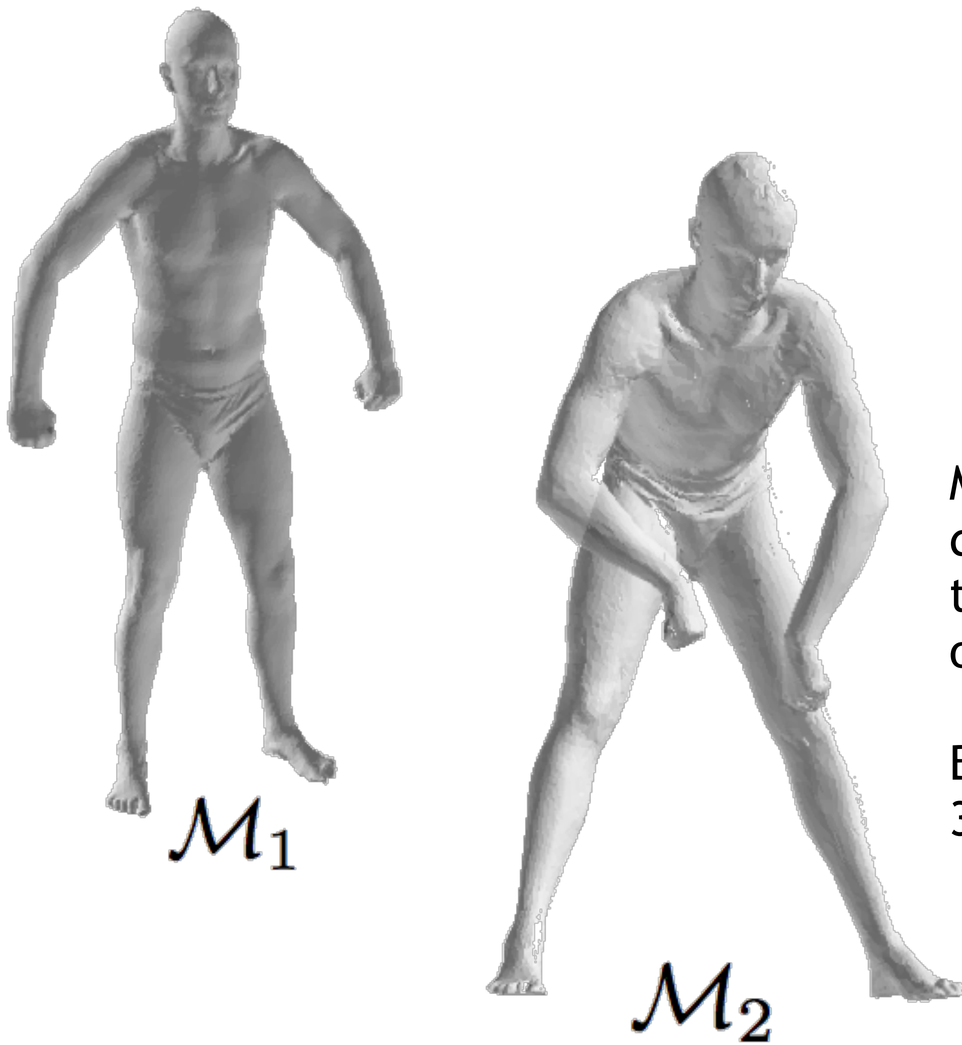
Graph Isomorphism



Intrinsic alignment of **manifolds**



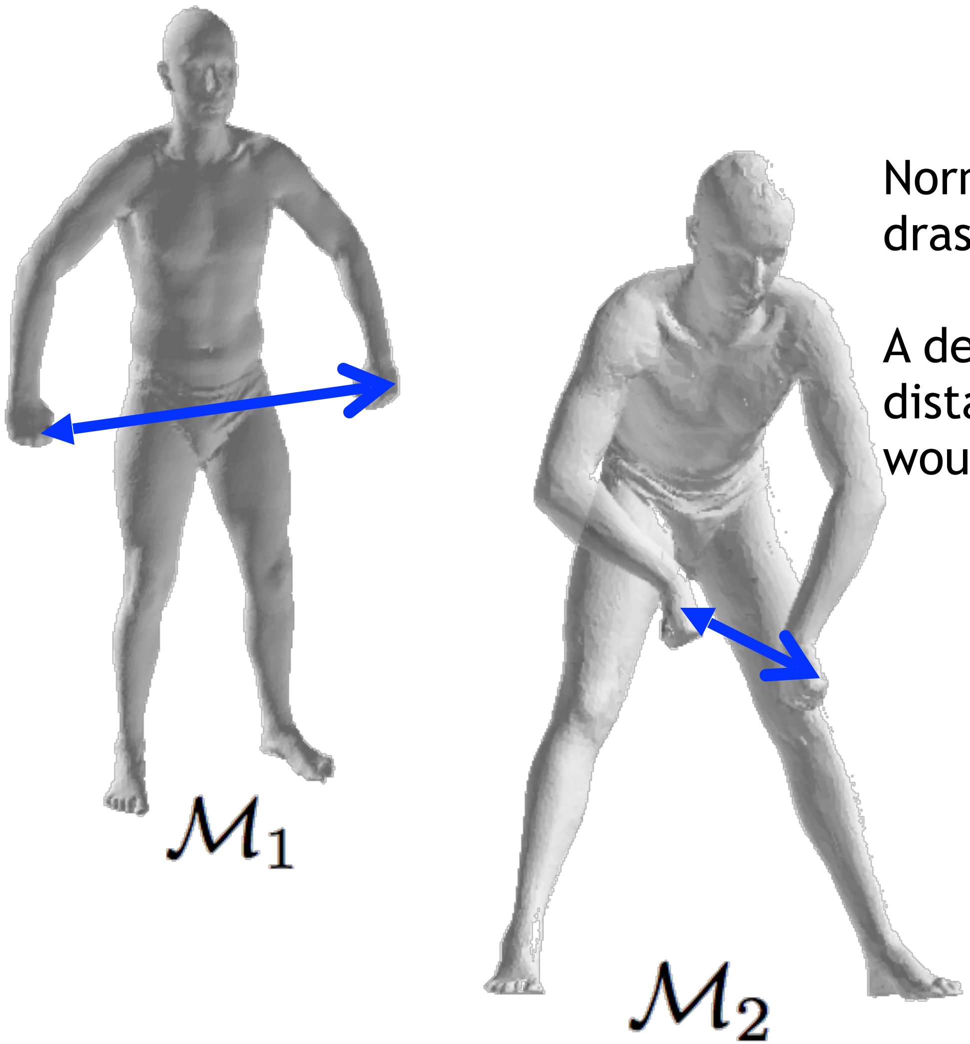
Why Intrinsic?



Many shapes have natural deformations and articulations that do not change the nature of the shape.

But they change its embedding 3D space.

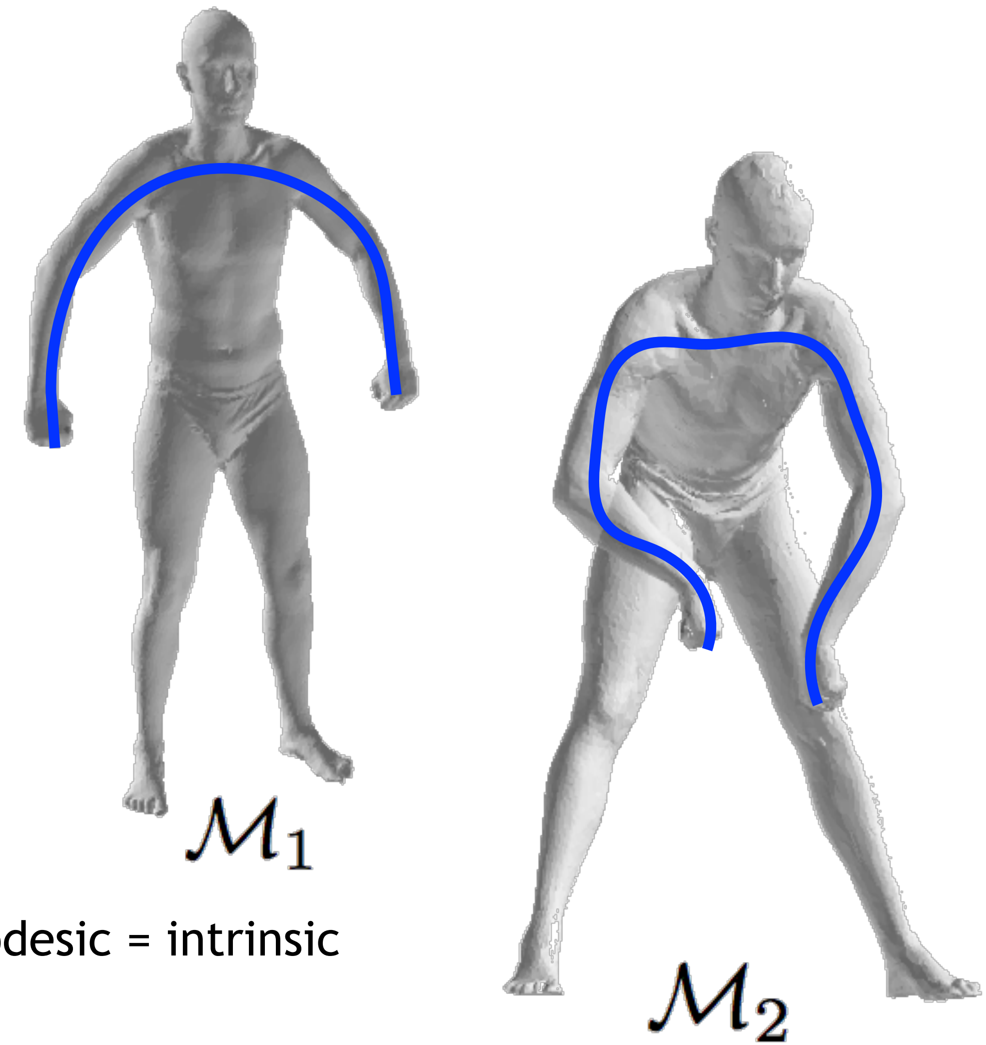
Why Intrinsic?



Normal distances can change drastically under such deformations

A descriptor based on Euclidean distance histograms, like D2, would fail

Geodesic / Intrinsic Distances



Near isometric deformations are common for both organic and man-made shapes

Intrinsic distances are invariant to isometric deformations

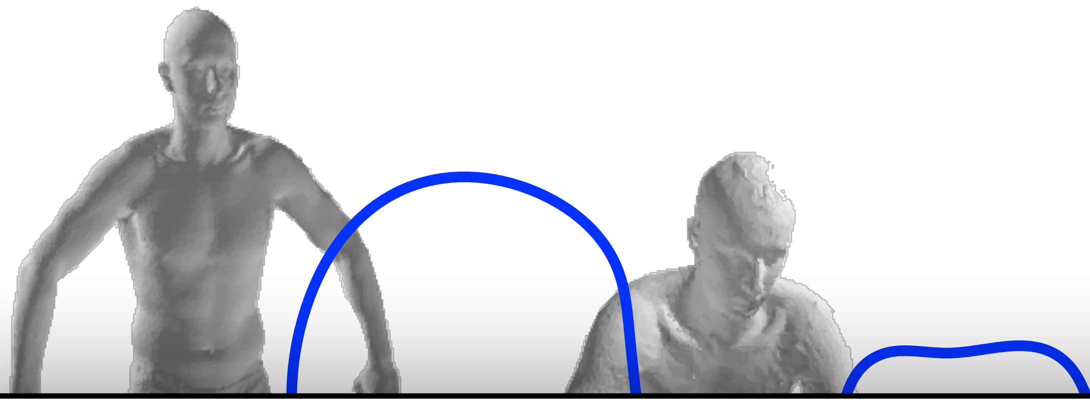
geodesic = intrinsic

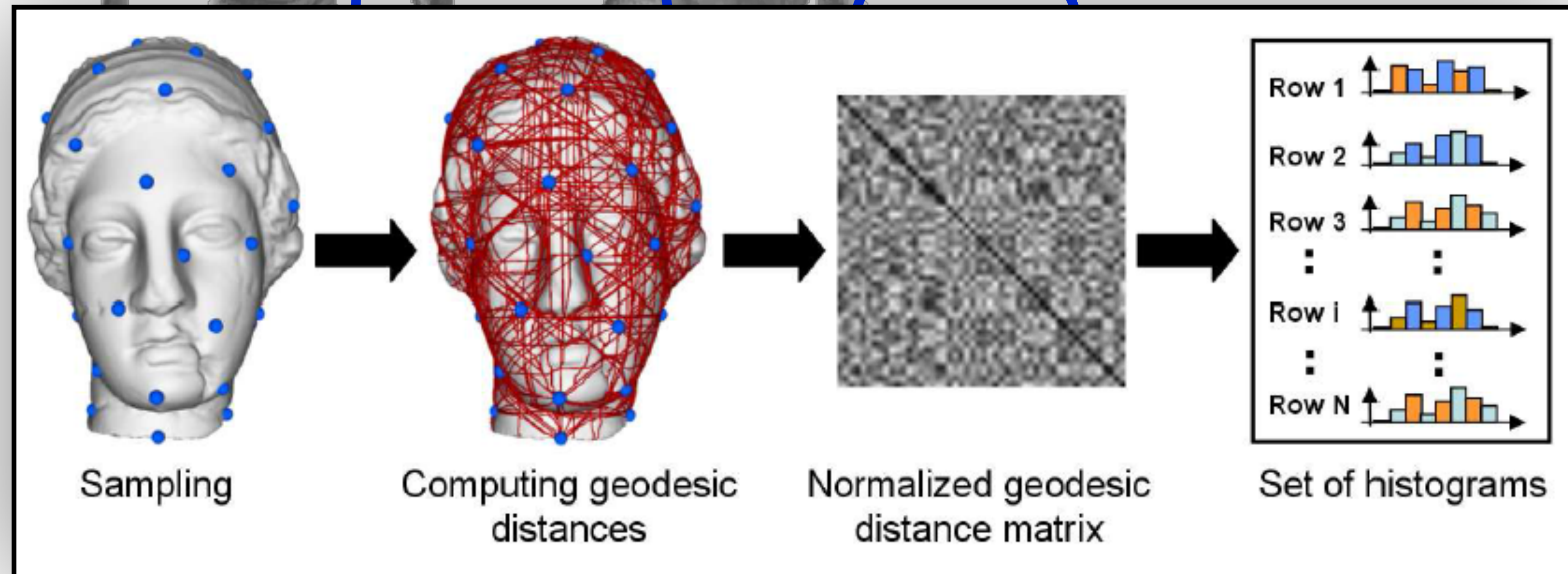
isometry = length-preserving transform

No stretching, shrinking, or tearing

Geodesic / Intrinsic Distances

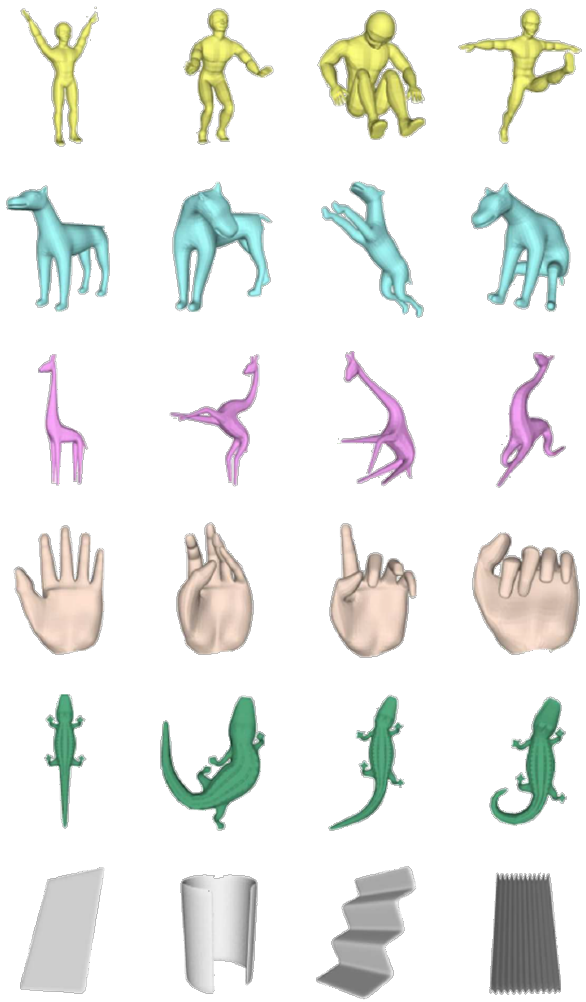
We can use geodesic distance histograms

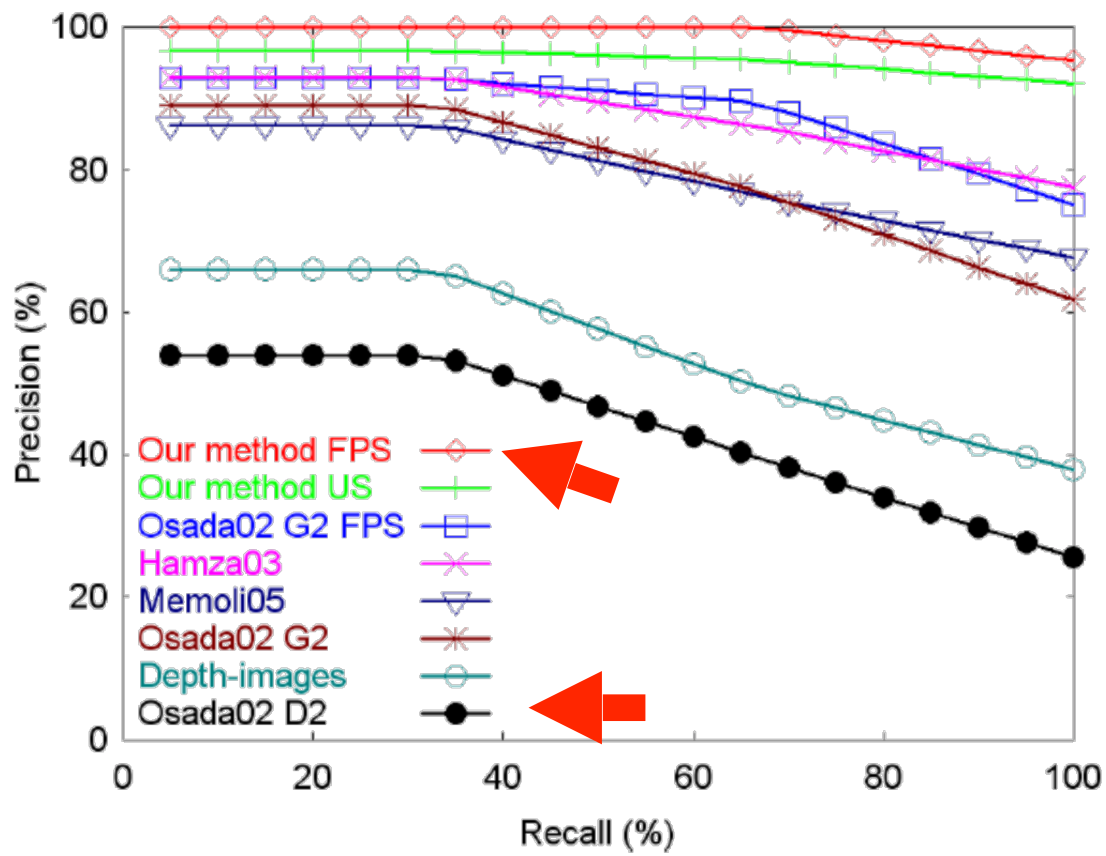




Ruggeri et al. 2008

Geodesic / Intrinsic Distances

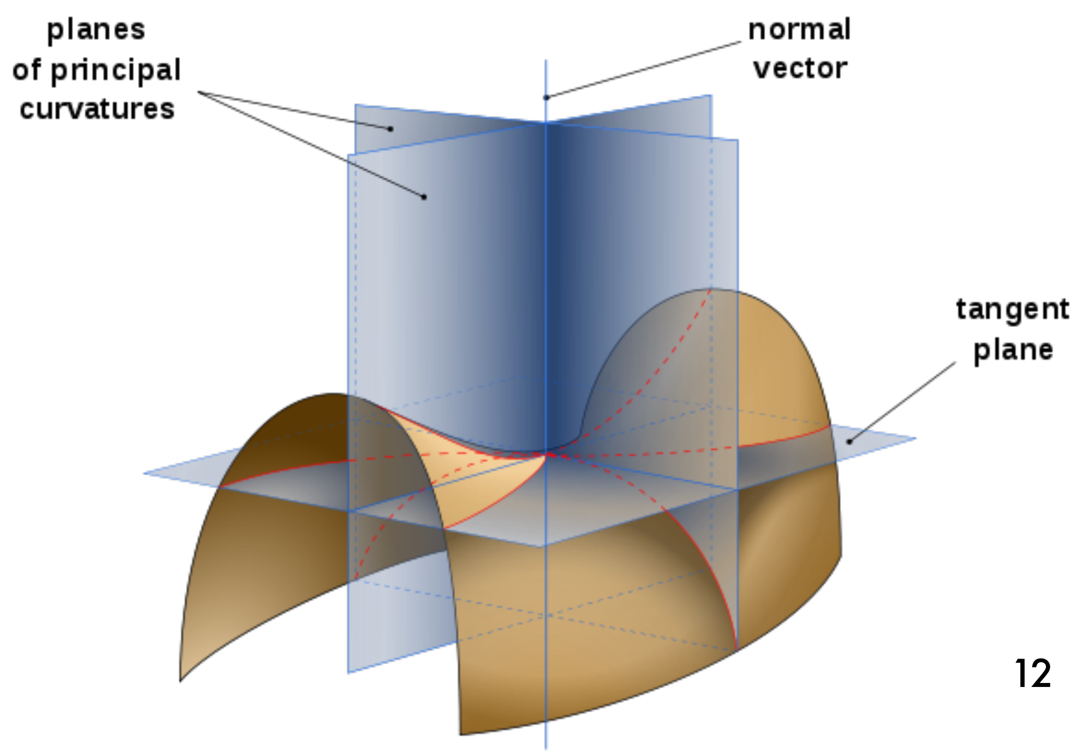




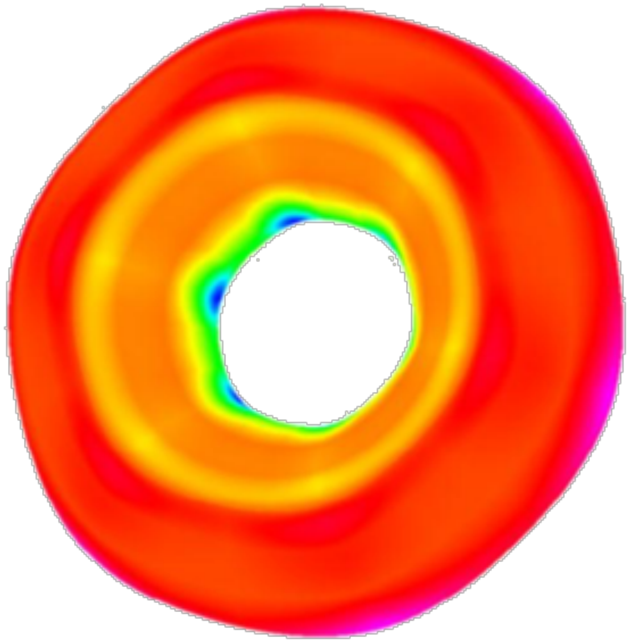
Ruggeri et al. 2008

What About Local Intrinsic Descriptors?

- Isometrically invariant features
  - Curvature
  - Geodesic Distance
  - Histogram of Geodesic Distances (similar to D2)
  - Global Point Signature
  - Heat Kernel Signature
  - Wave Kernel Signature



Gaussian Curvature



$$K = \kappa_1 \kappa_2$$

Theorema Egregium

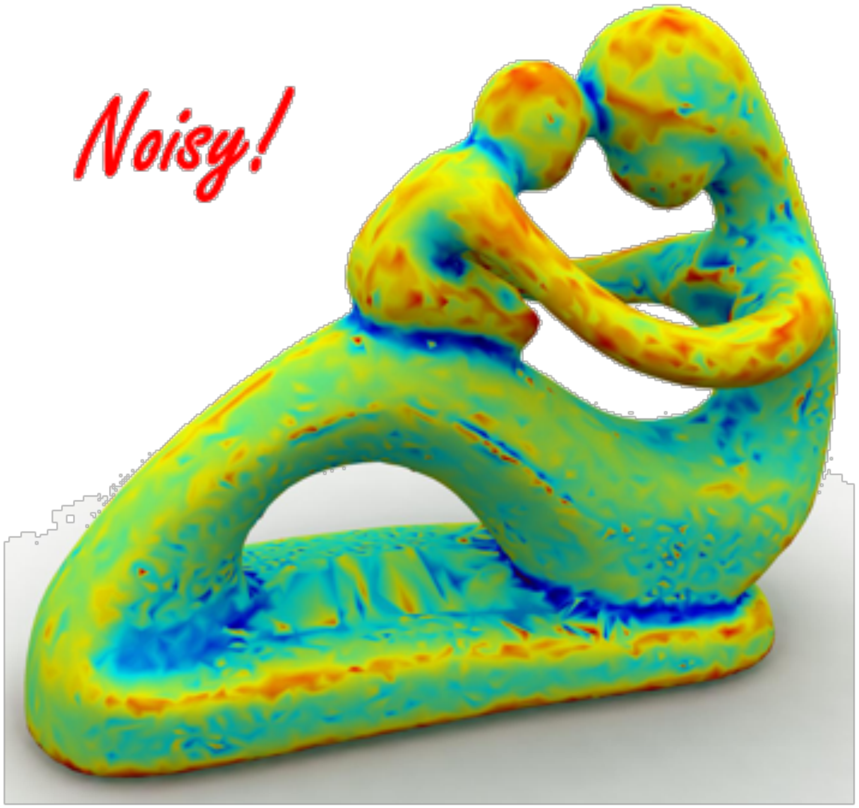
("Remarkable Theorem"):

Gaussian curvature is intrinsic.

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian Curvature

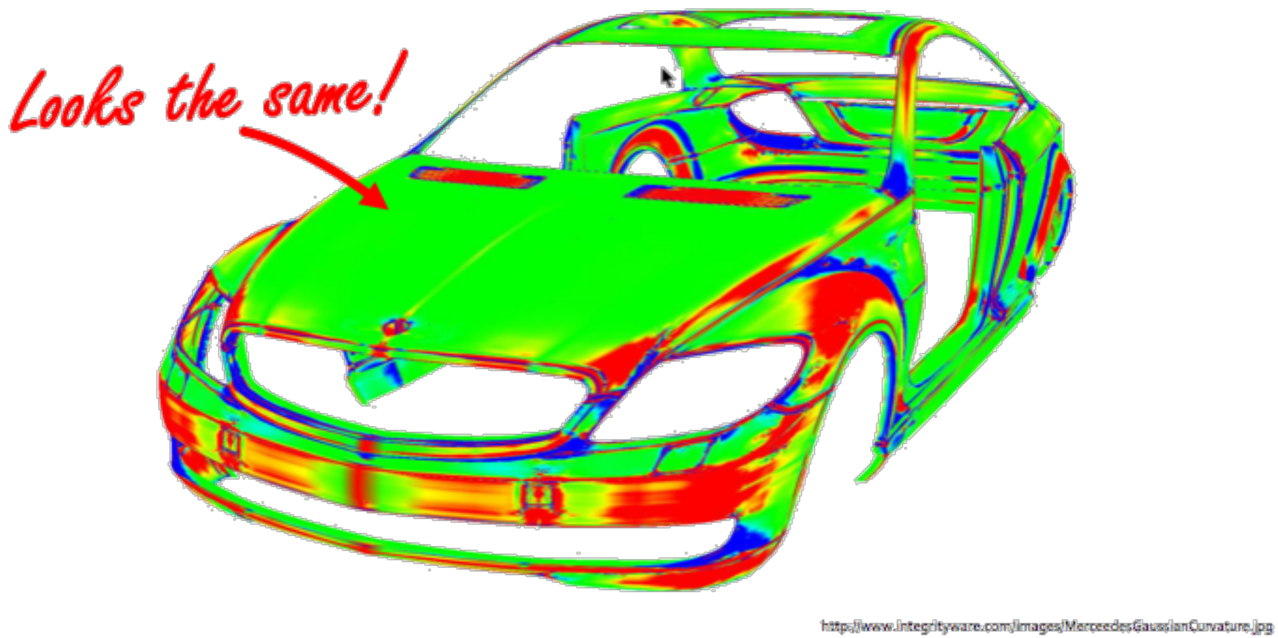
Problems



$$K = \kappa_1 \kappa_2$$

Gaussian Curvature

Problems



$$K = \kappa_1 \kappa_2$$

Solomon

Spectral Intrinsic Signatures

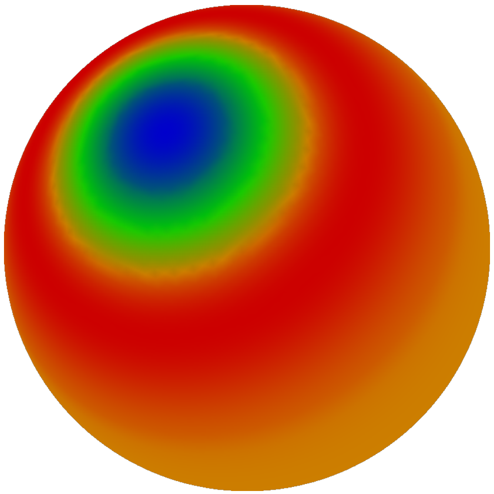
Laplace-Beltrami Operator

- Analog of Fourier transform on the sphere, but now on a general 2D manifold
- LB is an operators that can be applied to functions on manifolds to yield other functions

$$\Delta : C^{\infty}(M) \to C^{\infty}(M), \Delta f = \text{div } \nabla f$$

$$\text{div}\nabla f$$

$$\frac{\partial f}{\partial t} = \Delta f = \text{div } \nabla f$$



LB Eigen-decomposition

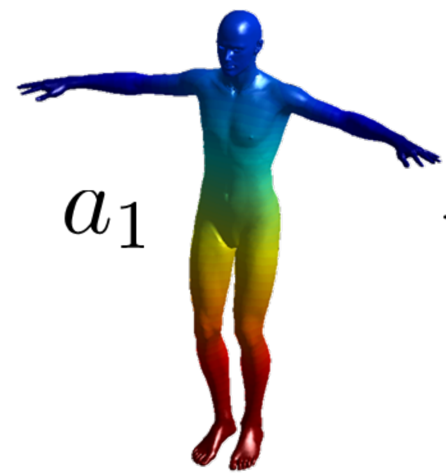
- The Laplace-Beltrami operator $\triangle$ has an eigendecomposition

$$\Delta \phi_i = \lambda_i \phi_i$$

$\lambda_0 = 0$ $\lambda_1 = 2.6$ $\lambda_2 = 3.4$ $\lambda_3 = 5.1$ $\lambda_4 = 7.6$

Multiscale Basis for a Function Space

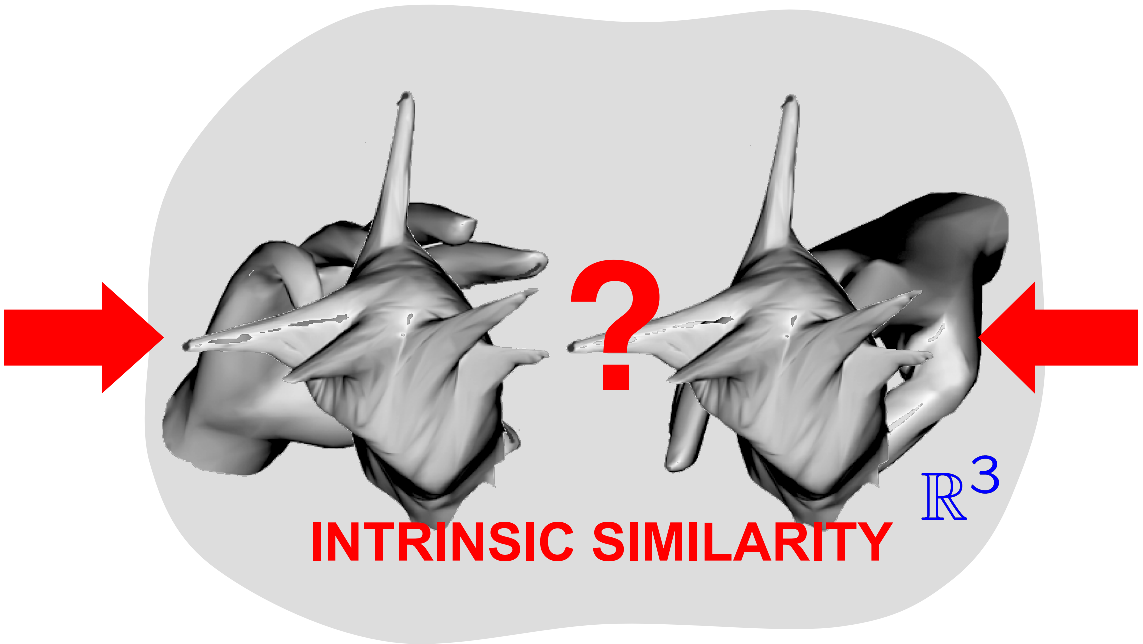
$f : M \to \mathbb{R}$

$= \quad a_0$  $+ \quad a_1$  $+ \quad \cdots$

$$f = \sum_{i=0}^{\infty} a_i \phi_i \qquad a_i = \int_M f(x)\phi_i(x)d\mu$$

Global Point Signature

$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

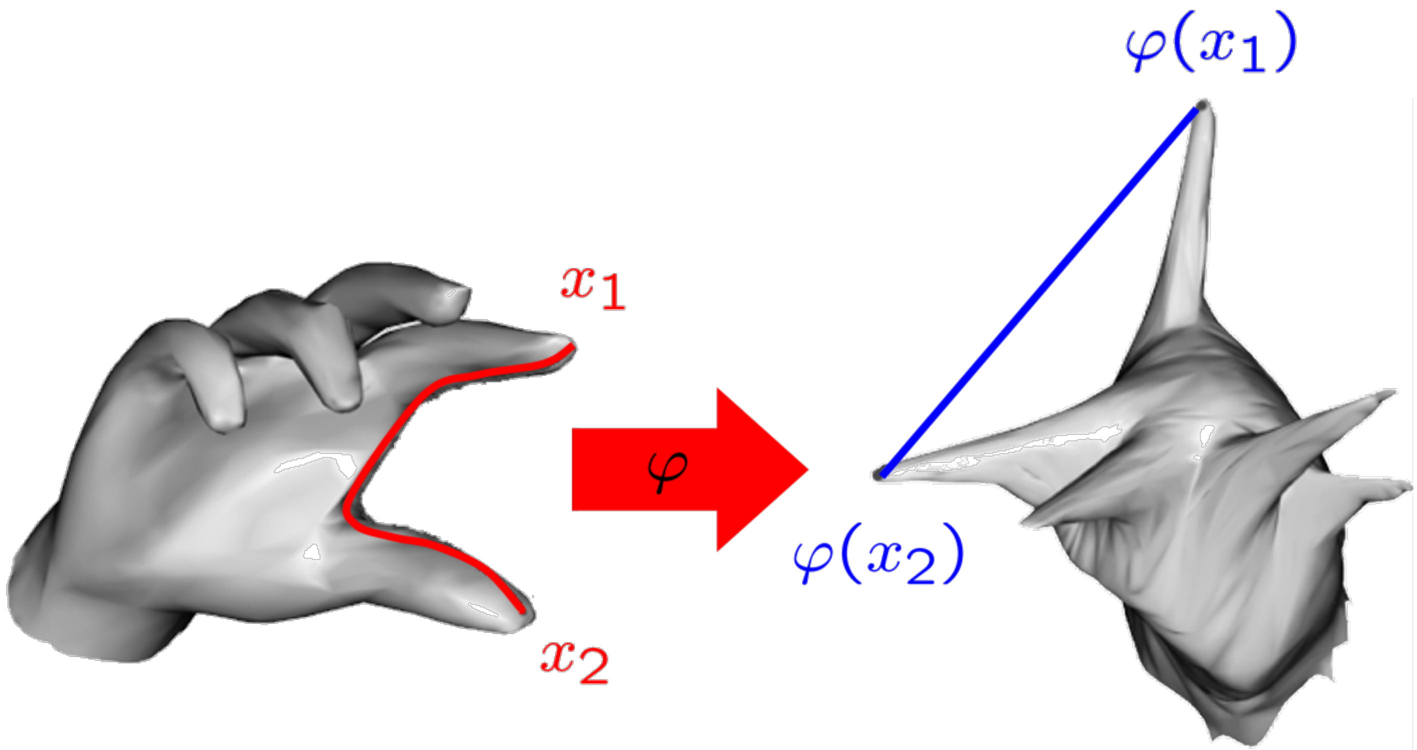


Rustamov et al. 2007

Global Point Signature

almost invariant under isometries – but not completely canonical

$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$



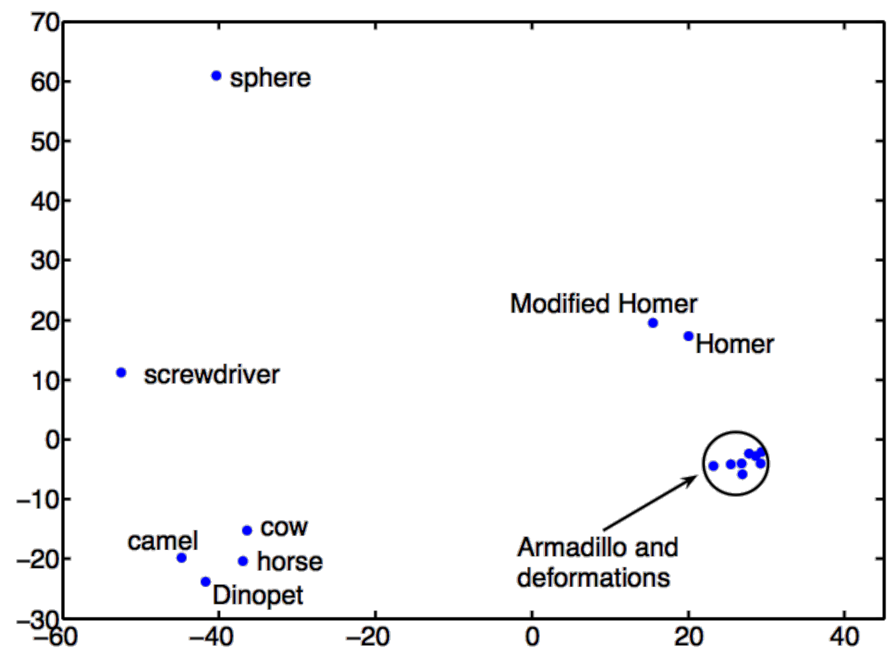
Diffusion distances are also intrinsic
and also canonical

Rustamov et al. 2007

Global Point Signature

$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \cdots \right)$$

Figure 4: *Armadillo and its deformations.*



Similar to D2, but use histograms in embedded space (rather than Euclidean)

Rustamov et al. 2007

Global Point Signature

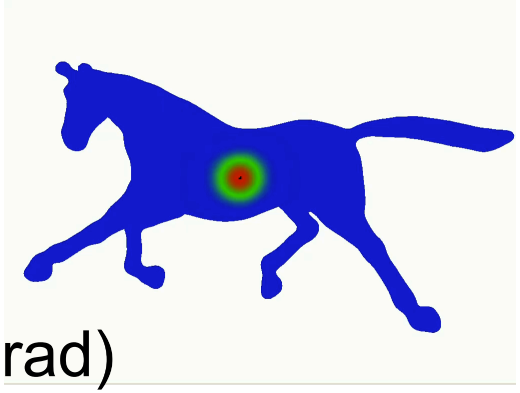
$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

- Pros
  - Isometry-invariant
  - Global (each point feature depends on entire shape)
- Cons
  - Eigenfunctions may flip sign
  - Eigenfunctions might change positions due to deformations
  - Only global

Rustamov et al. 2007

Back to Heat Diffusion

- Heat diffusion on a Riemannian manifold:
  - If $u(x,t)$ is the amount of heat at point x at time t, then



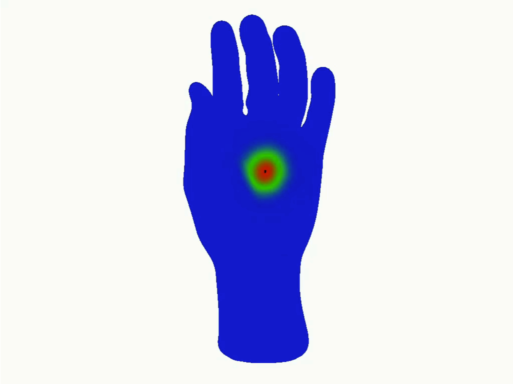
$$\frac{\partial u}{\partial t} = \Delta u$$

  - Δ : Laplace-Beltrami Operator (div grad)

- Given an initial distribution $f(x)$ After time t

$$f(x,t) = e^{-t\Delta} f$$

H_t heat operator

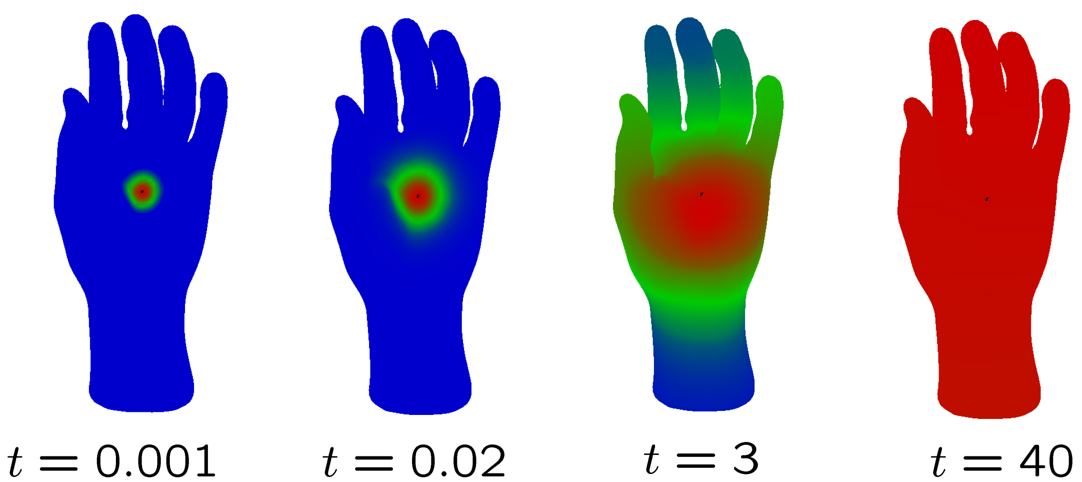


The Heat Kernel

- Heat kernel $k_t(x, y)$:

$$f(x,t) = \int_{\mathcal{M}} k_t(x,y) f(y) dy$$

$k_t(x, y)$: amount of heat transferred from x to y in time t. How well x and y are connected at scale t



Background

- Heat Kernel $k_t(x, y)$. Also the probability density function of Brownian motion on $\mathcal{M}$:

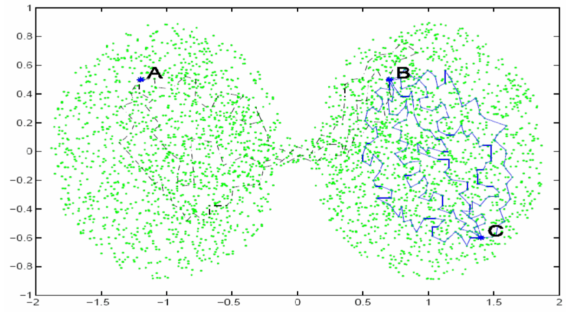
$$\mathbb{P}\left(W_x^t \in C\right) = \int_C k_t(x, y) dy$$

- Intuitively: weighted average over all paths possible between x and y in time t

- Related to Diffusion Distance:

$$D_t(x, y) = k_t(x, x) - 2k_t(x, y) + k_t(y, y)$$

- a robust multi-scale measure
- of proximity



Coifman, Lafon

Heat Kernel Properties

Basic Properties

- $k_t(x, y) = k_t(y, x)$
- $k_{t+s}(x, y) = \int_M k_t(x, z) k_s(z, y) dz$
- $k_t(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$

Eigenfunctions of LB

Heat Kernel Properties

- Invariant under isometric deformations

  If $T : X \to Y$ is an isometry, then:

  $$k_t(X, Y) = k_t(T(x), T(y))$$

- Conversely: it characterizes the shape up to isometry.

  If $k_t(X, Y) = k_t\left(T(x), T(y)\right) \;\forall\; x, y, t$ then:
  T is an isometry.

  This is because:

  $$\lim_{t \downarrow 0} \left(t \log k_t(x, y)\right) = -\frac{1}{4} d^2_{\mathcal{M}}(x, y) \;\forall\; x, y$$

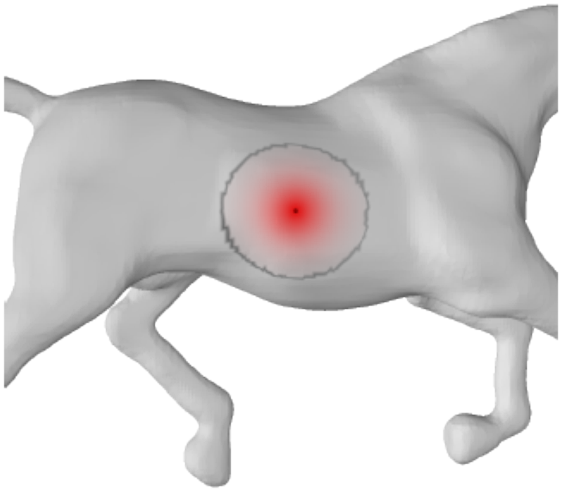
  where $d_{\mathcal{M}}(\cdot, \cdot)$ is the geodesic distance

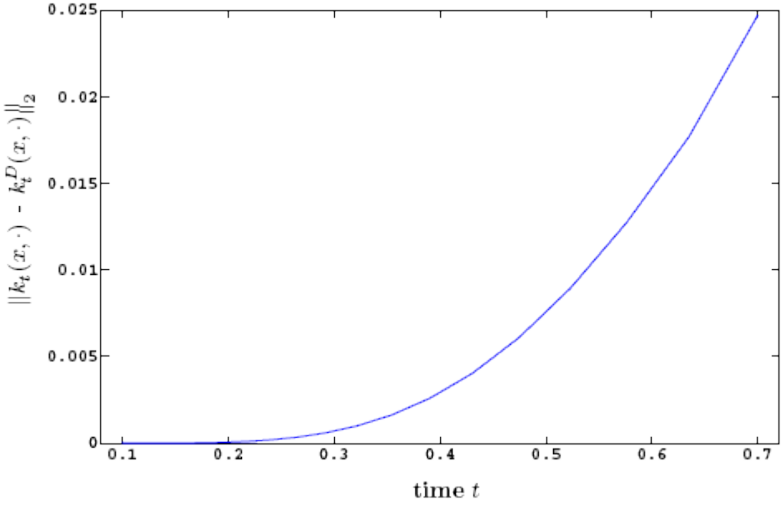
Heat Kernel Properties

- Multiscale:

  For a fixed x, as t increases, heat diffuses to larger and larger neighborhoods

  Therefore, $k_t(x, \cdot)$ is determined by (reflects the properties of) a neighborhood that grows with t

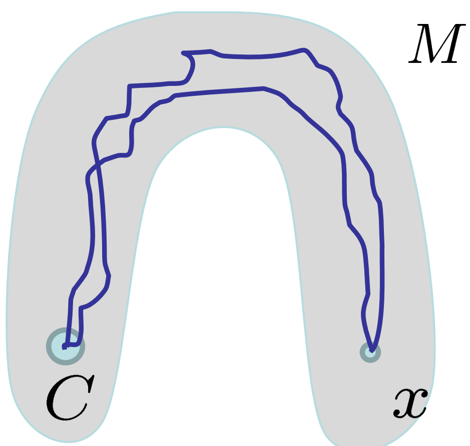




Heat Kernel Properties

- Robustness:

  $k_t(x,\cdot)$ is the probability density function of BM, a weighted average over all paths, which is generally not very sensitive to local perturbations

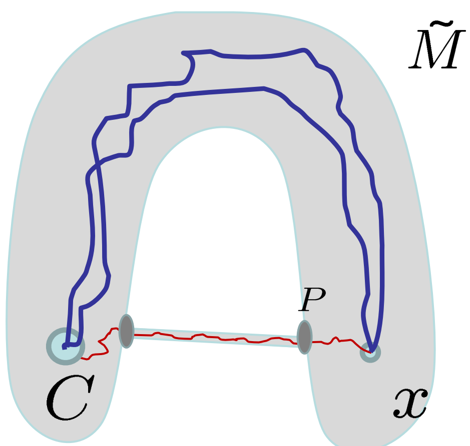


$$k_t^M(x, C) = \mathbb{P}(W_x^t \in C)$$

Heat Kernel Properties

- Robustness:

  $k_t(x, \cdot)$ is the probability density function of BM, a weighted average over all paths, which is generally not very sensitive to local perturbations



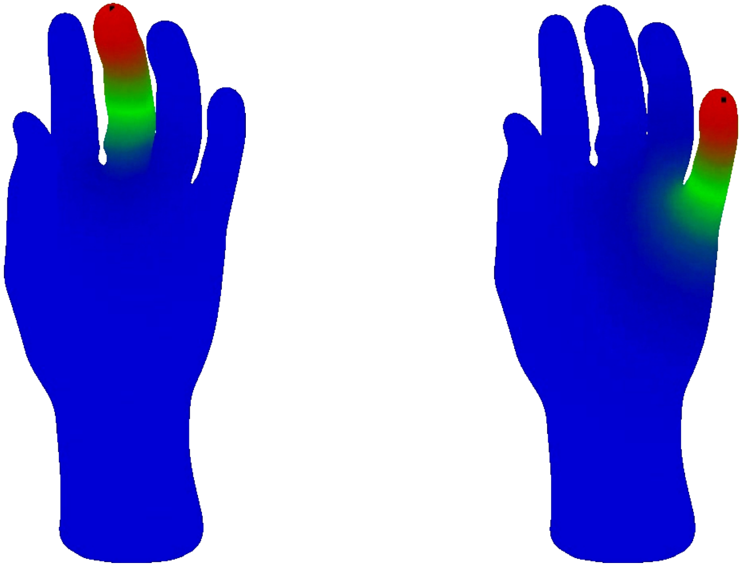
$$k_t^{\tilde{M}}(x, C) = \mathbb{P}(\tilde{W}_x^t \in C)$$

Only paths through the modified area P will change

Defining a Signature

- Let $k_t(x, \cdot)$ be the signature of x at scale t

  The heat kernel has all the properties we want Except easy comparison …



- $k_t(x, \cdot)$ is a function on the entire manifold
- Nontrivial to align the domains of such functions across different shapes, or even for different points of the same shape

The Heat Kernel Signature

- Let $k_t(x, \cdot)$ be the signature of x at scale t

  The heat kernel has all the properties we want.

  Except easy comparison …

- We define the Heat Kernel Signature (HKS), by restricting to the diagonal:

$$\mathrm{HKS}(x) = \{k_t(x, x), t \in \mathbb{R}^+\}$$

- Now HKSs of two points can be easily compared since they are defined on a common domain (time)

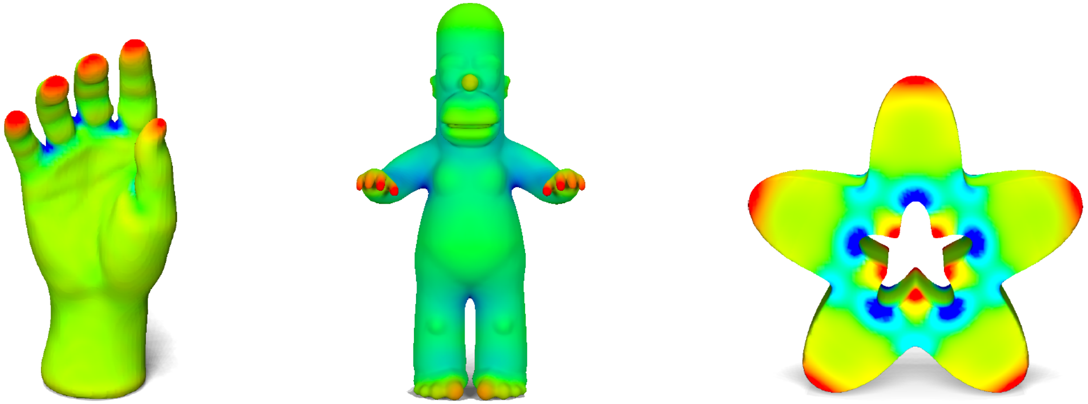
[Sun, Ovsjanikov, G., 2009]

Defining a Signature

- Since HKS is a restriction of the heat kernel, it is:
  - Robust
  - Multiscale

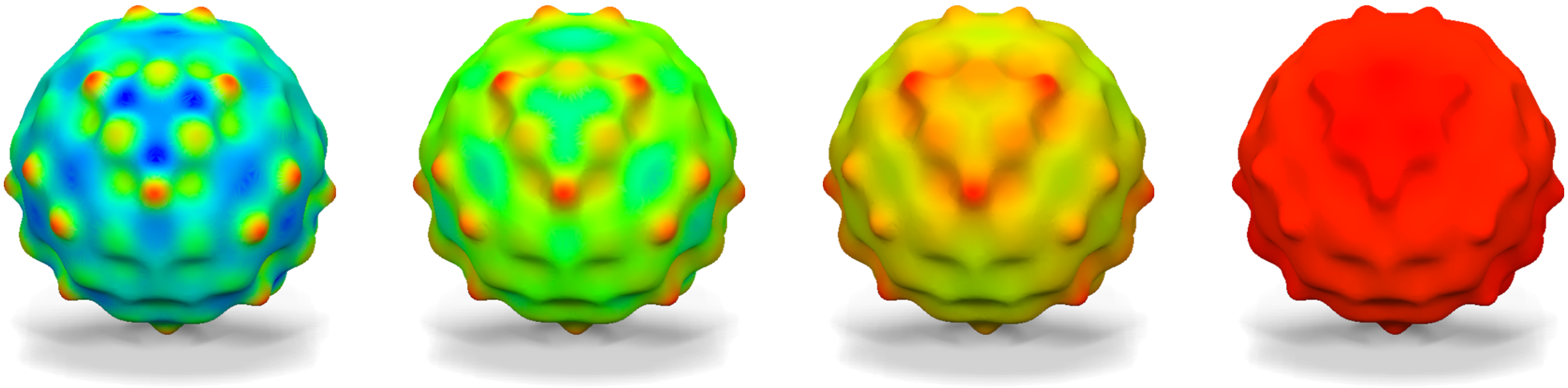
- Question: How informative is it?
  - Related to Gaussian curvature for small t:

$$k_t(x,x) = \frac{1}{4\pi t}\sum_{i=0}^{\infty} a_i t^i \qquad a_0 = 1, a_1 = \frac{1}{6}K$$



Defining a Signature

- HKS can be interpreted as a multiscale, robust, intrinsic curvature:



$t = 0.004$ $t = 0.008$ $t = 0.02$ $t = 2$

Informative Theorem

- The set of all HKSs on a shape almost always defines it up to isometry!

- Theorem: If X and Y are two compact manifolds, such that Δ_X and Δ_Y have only non-repeating eigenvalues, then a homeomorphism $T : X \to Y$ is an isometry if and only if, for all **x**

$$\mathsf{HKS}(x) = \mathsf{HKS}(T(x))$$

- The set of all HKSs characterizes the intrinsic structure of the manifold

Informative Theorem

- Intuition: Heat kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

$$\mathrm{HKS}(x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x)$$

- If eigenvalues do not repeat, we can recover $\{\lambda_i\}$ and $\{\phi_i^2(x)\}$ from $\mathrm{HKS}(x)$. E.g. $\lambda_0 = 0$

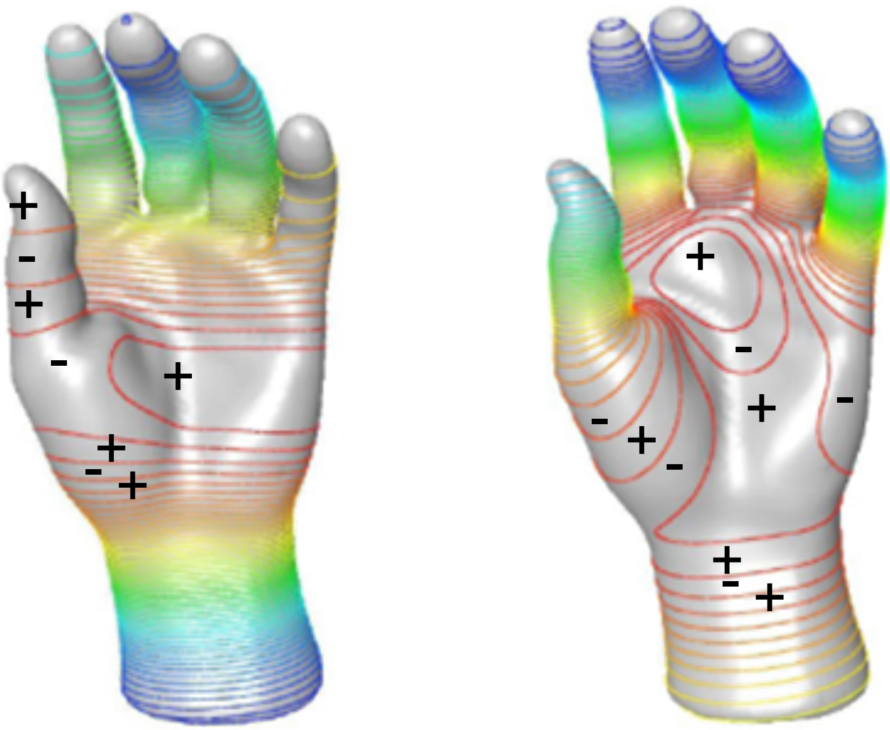
$$\phi_0^2(x) = \lim_{t\downarrow 0} \mathrm{HKS}(x,t)$$

and

$$\lambda_1 = \inf\left\{a \text{ s.t. } \lim_{t\downarrow 0} e^{at}(\mathrm{HKS}(x,t) - \phi_0^2(x)) \neq 0\right\}$$

Informative Theorem

- Nodal domains of the eigenfunctions of LB: domains that are delimited by the zeroes of an eigenfunction



Key property: they are sign interleaved:

No two domains of the same sign can border each other

Note that any mapping that preserves squared values must map a nodal domain to another. Moreover, by fixing a sign of one point, the signs of all other points are fixed by continuity

Informative Theorem

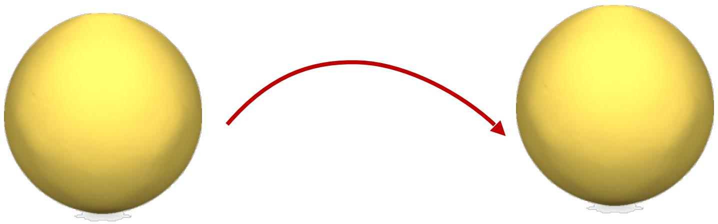
- Intuition: Heat Kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

$$\text{HKS}(x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x)$$

- After recovering the eigenvalues, and squared eigen-functions, we know that $\left|\phi_i^Y(T(x))\right| = \left|\phi_i^X(x)\right|$
- We use the properties of nodal domains of eigen-functions to show: $\phi_i^Y(T(x)) = \phi_i^X(x)$ or $\phi_i^Y(T(x)) = -\phi_i^X(x)$
- Since the eigenvalues + eigenfunctions define the manifold, the theorem follows

Informative Theorem

- How general is the theorem?
  - If there are repeated eigenvalues, it does not hold:



On the sphere, $\mathrm{HKS}(x) = \mathrm{HKS}(y) \ \forall \ x, y$ but there are non-isometric maps between spheres.

- Uhlenbeck's Theorem (1976): for "almost any" metric on a 2-manifold X, the eigenvalues of Δ_X are non-repeating

Informative Property

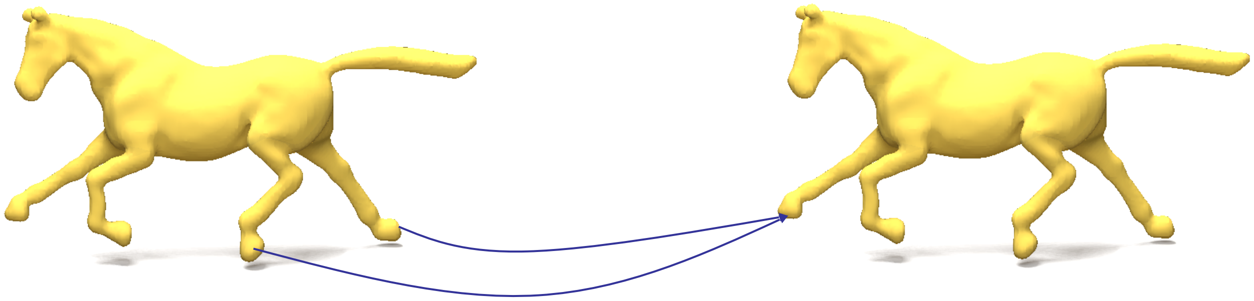
- Conclusion:
  - HKS is informative for individual points
  - And, as a set, for the entire shape

Can be used both for multiscale point matching and for shape comparison

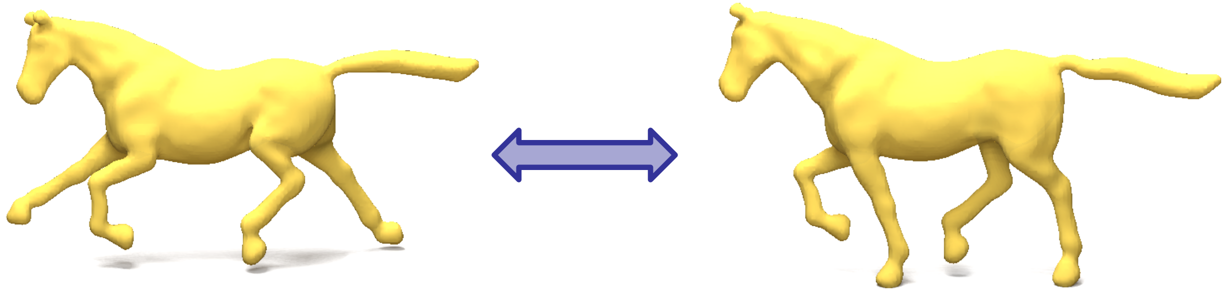
$$\mathrm{HKS}(x) = \{k_t(x,x), t \in \mathbb{R}^+\}$$

Applications

- Multi-scale matching with HKS, structure discovery

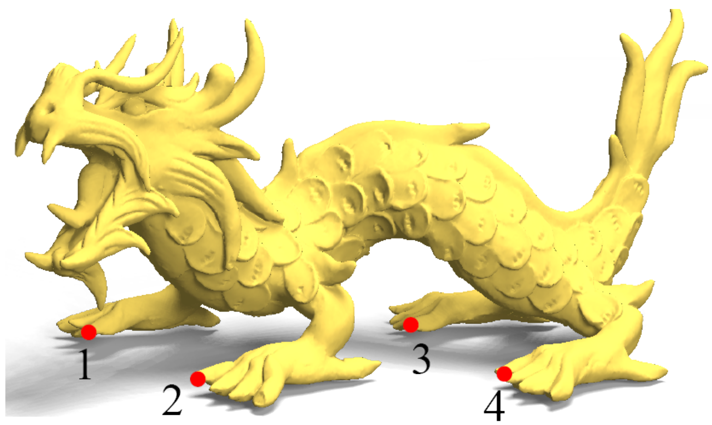


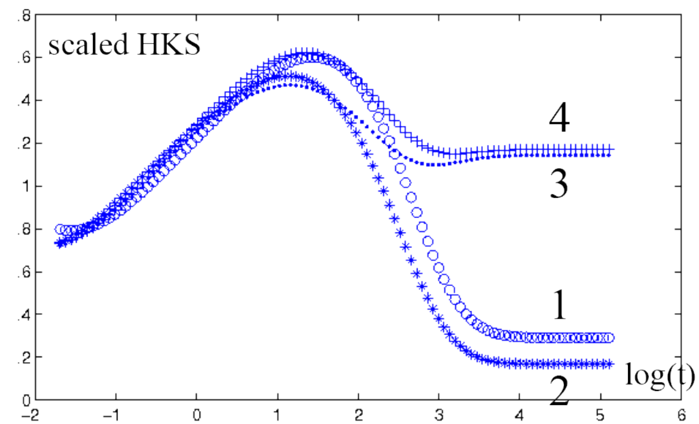
- Shape comparison
  - Shape retrieval using HKS
  - Spectral version of Gromov-Hausdorff

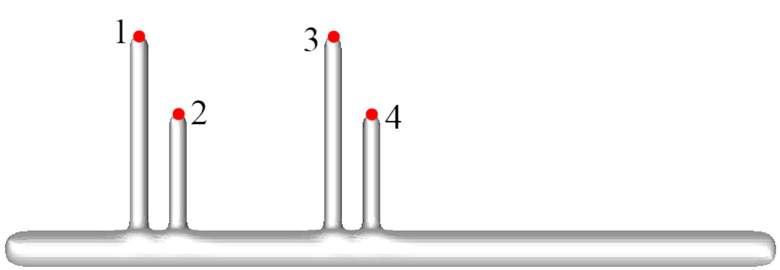


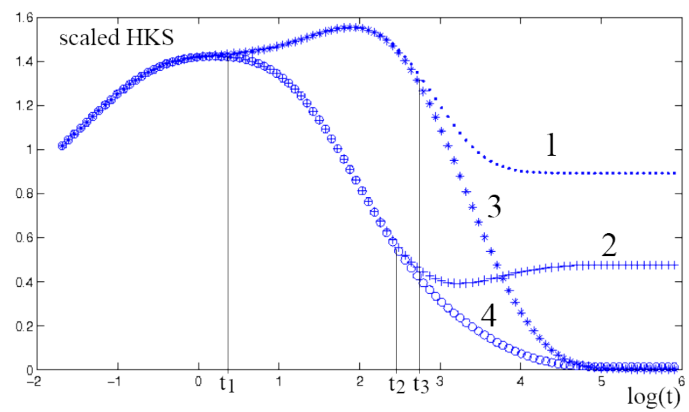
Multiscale Matching

- Comparing points through their HKS signatures:





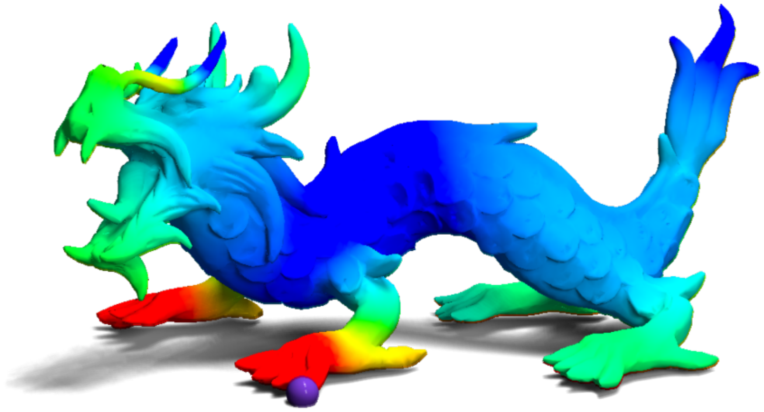




Multiscale Matching

- Comparing points through their HKS signatures:

Medium scale



Full scale

Multiscale Matching

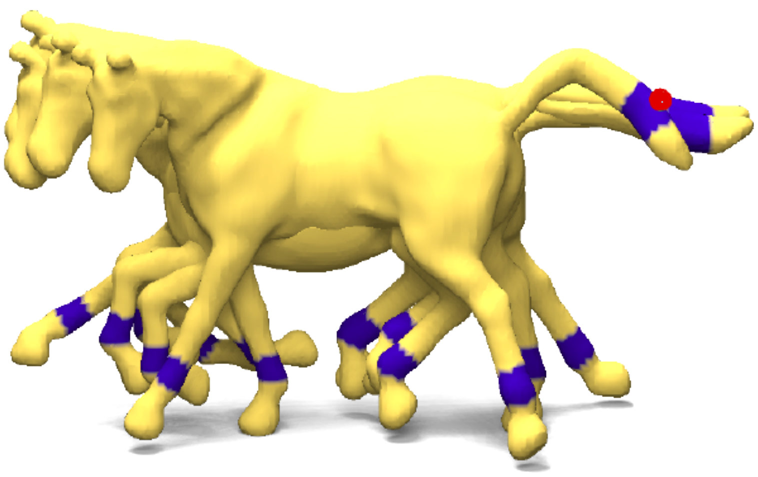
- Finding similar points – robustly:

Medium scale

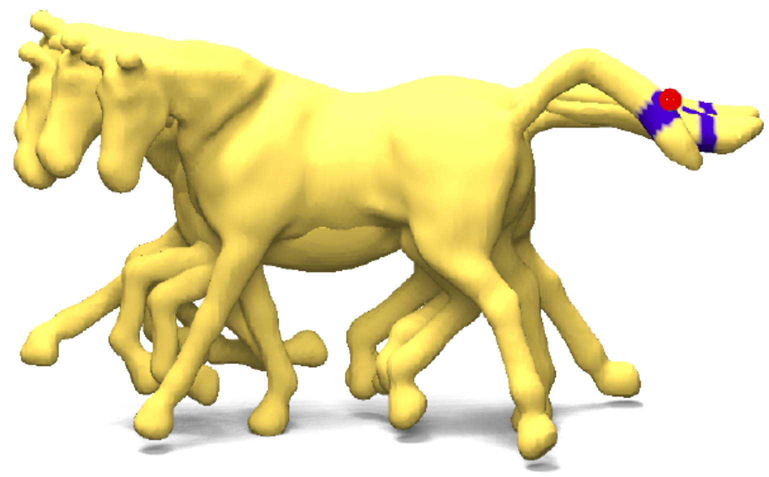
Full scale

Multiscale Matching

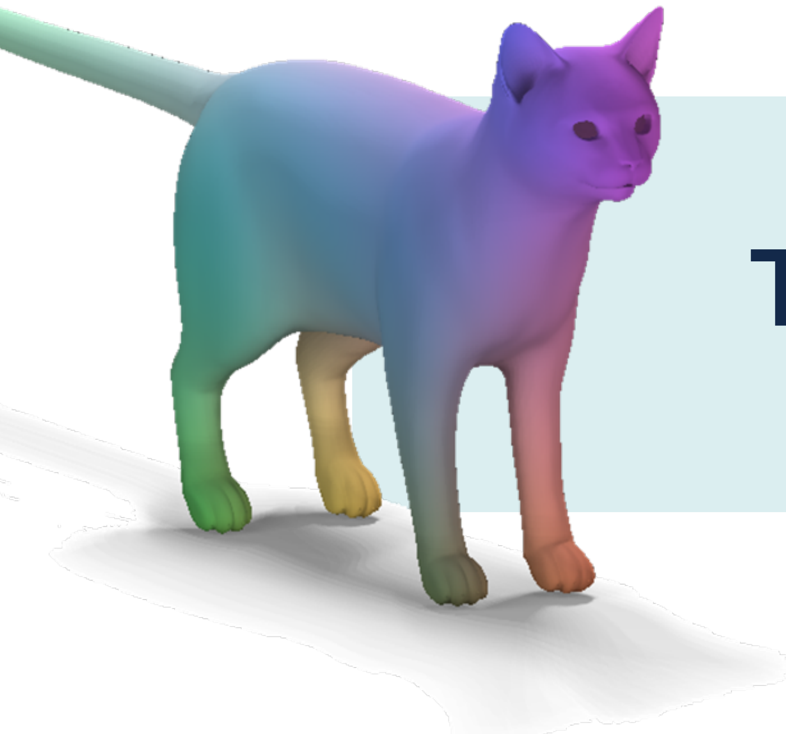
- Finding similar points across multiple shapes:



Medium scale



Full scale



The End