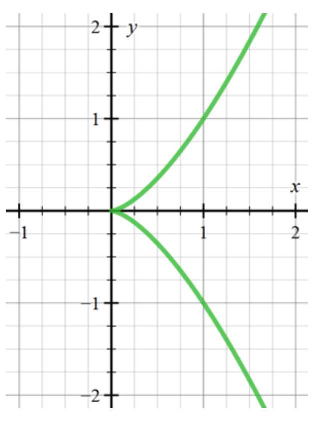
Lecture 3: Curves

- Cusp

On P4 of the slides, we showed the following curve that has a "cusp" point:



While the curve is not smooth, it can be parameterized by smooth functions. For example, we can parameterize the trajectory by (t^2, t^3) by tracing t from $-\infty$ to $+\infty$.

So one may ask: *What is the condition that the parameterized curve is smooth?*

More discussions over this problem can be found from Wikipedia. Generally speaking, cusps may occur when the gradient vanishes at the point.

- Proof of $\|T(s)\| = 1$, where $T(s)$ is the tangent of a curve at s under arc-length parameterization.

Define $s(t)$ to be the length of curve at the interval $[0, t]$, then $s(t) = \int_0^t \|\gamma'(t)\| dt$.

Define $\tilde{\gamma}(s) = \gamma(t(s))$, where $t(s)$ is the inverse function of $s(t)$.

Then $T(s) = \frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt}\frac{dt}{ds} = \gamma'(t)\frac{dt}{ds}$. But $\frac{dt}{ds} = 1/\frac{ds}{dt} = \frac{1}{\|\gamma'(t)\|}$. So $\|T(s)\| = 1$.

- Proof of $\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$.

First, by $\|T(s)\|^2 = 1$, $\frac{dT(s)^T T(s)}{ds} = 2T(s)\frac{dT(s)}{ds} = 0$. So $T(s) \perp \frac{dT(s)}{ds}$.

By the definition of normal vector $N(s) = JT(s)$, which is a unit vector that rotates $T(s)$ counter-clockwise by 90°, $N(s)$ is paralell to $\frac{dT(s)}{ds}$. So we introduce the curvature $\kappa(s)$ to be the coefficient $\frac{dT(s)}{ds} = \kappa(s)N(s)$. By the unity of $N(s)$, we can also use $\kappa(s) = \langle \frac{dT(s)}{ds}, N(s)\rangle$ to compute $\kappa(s)$.

Next we show that $\frac{dN(s)}{ds} = -\kappa(s)T(s)$.

$\frac{dN(s)}{ds} = \frac{dJT(s)}{ds} = J\frac{dT(s)}{ds} = J[\kappa(s)N(s)] = \kappa(s)JJT(s) = -\kappa(s)T(s)$, because $JJ = -I$.

- Example: Curvature of a circle

Assume C a circle of a radius r whose center is at the origin. We can parameterize C as $\gamma(\theta) = (r\cos\theta, r\sin\theta)$.

The arc-length of the circle can be calculated as

$$s(\theta_0) = \int_0^{\theta_0} \|\gamma(\theta)'\| d\theta = r\theta_0$$

So the arc-length parameterization of the circle is

$$\gamma(s) = (r\cos(s/r), r\sin(s/r))$$

The tangent vector is $T(s) = \dot{\gamma}(s) = (-\sin(s/r), \cos(s/r))$. We can verify that $\|\dot{\gamma}(s)\| = 1$.

To compute the curvature, we take the derivative of $T(s)$:

$$\frac{dT(s)}{ds} = (-\frac{\cos(s/r)}{r}, -\frac{\sin(s/r)}{r})$$

Since $N(s) = JT(s) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\sin(s/r) \\ \cos(s/r) \end{bmatrix} = \begin{bmatrix} -\cos(s/r) \\ -\sin(s/r) \end{bmatrix}$, we can tell that $\kappa(s) = \frac{1}{r}$ by $\frac{dT(s)}{ds} = \frac{1}{r}N(s)$.

- Frenet frame

Denote the coordinate transformation matrix as F.

By $\|T\|^2 = 1$, we derive that $\frac{dT(s)}{ds} \perp T(s)$. Similarly, we can derive that $\frac{dN(s)}{ds} \perp N(s)$ and $\frac{dB(s)}{ds} \perp B(s)$. That explains why the diagonals are 0's: $F_{11} = F_{22} = F_{33} = 0$.

Because $\|T(s)\| = 1$, we define $T(s)$ to be the first basis of the local frame at s. By the fact that $\frac{dT(s)}{ds} \perp T(s)$, we can define the second basis as the unit vector along $\frac{dT(s)}{ds}$, which is $N(s) = JT(s)$ that rotates $T(s)$ by 90° in the plane defined by $T(s)$ and $\frac{dT(s)}{ds}$. Similar as the 2D case, we introduce $\kappa(s) = \langle \frac{dT(s)}{ds}, N(s)\rangle$. So $F_{12} = \kappa(s)$. Later we will see that B is perpendicular to N, so $F_{13} = 0$.

Taking the derivative of $N(s)$ and $\frac{dN(s)}{ds}$ becomes a new vector that may not be spanned in $\{T(s), N(s)\}$. Note that $\langle \frac{dN(s)}{ds}, T(s)\rangle = \langle N(s), T(s)\rangle - \langle N(s), \frac{dT(s)}{ds}\rangle = -\kappa(s)$. So $F_{21} = -\kappa(s)$. Suppose that $\frac{dN(s)}{ds}$ has a component that is perpendicular to the $\{T(s), N(s)\}$ plane. Denote the unit vector of this component by $B(s)$ and treat it as the third basis of the Frenet frame. Then we let $\tau(s) = \langle \frac{dN(s)}{ds}, B(s)\rangle$, i.e., $F_{23} = \tau(s)$.

$\langle \frac{dB(s)}{ds}, T(s)\rangle = \langle B(s), T(s)\rangle - \langle B(s), \frac{dT(s)}{ds}\rangle = -\langle B(s), \kappa(s)N(s)\rangle = 0$, so $F_{31} = 0$.

Also, $\langle \frac{dB(s)}{ds}, N(s)\rangle = \langle B(s), N(s)\rangle - \langle B(s), \frac{dN(s)}{ds}\rangle = -\tau(s)$ by the definition of $\tau(s)$, so $F_{32} = -\tau(s)$.

The general Frenet frame for $\gamma(s) : \mathbb{R} \to \mathbb{R}^n$

The above procedure actually gives a general algorithm to define a frame along a curve in arbitrary dimension:

Algorithm of General Frenet Frame

- Let $e_1(s) = T(s)$ be the first basis
- Generate the rest $n-1$ basis:
  1. Let $a(s)$=$\frac{de_{k-1}(s)}{ds}$
  2. Let $b(s)$ be the projection of $a(s)$ in the span of $\{e_1, \cdots, e_{k-1}\}$. Normalize the residual $a(s) - b(s)$ to be a unit vector and use it as the k–th basis e_k.
  3. Introduce $\chi_k(s) = \langle a(s), e_k(s)\rangle$
- The Frenet frame satisfies the relationship

$$\frac{d}{ds}\begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix} = \begin{pmatrix} 0 & \chi_1(s) & & 0 \\ -\chi_1(s) & \ddots & \ddots & \\ & \ddots & 0 & \chi_{n-1}(s) \\ 0 & & -\chi_{n-1}(s) & 0 \end{pmatrix}\begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix}$$

To verify the correctness, we just need to check the following:

1. $\|e_k\|^2 = 1$ implies that $\frac{de_k(s)}{ds}e_k(s) = 0$. So the diagonals are 0's.
2. $\langle \frac{de_k(s)}{ds}, e_{k-1}(s)\rangle = -\langle e_k(s), \frac{de_{k-1}(s)}{ds}\rangle = -\chi_{k-1}(s)$.
3. $\langle \frac{de_k(s)}{ds}, e_{k-i}(s)\rangle = -\langle e_k(s), \frac{de_{k-i}(s)}{ds}\rangle$. Note that $\frac{de_{k-i}(s)}{ds}$ lives in the space spanned by $\{e_1, e_2, \cdots, e_{k-i+1}(s)\}$, which is orthogonal to e_k when $i \geq 2$. So the rhs is 0.