UC San Diego

# Curves: Gauss Map, Turning Number Theorem, Parallel Transport

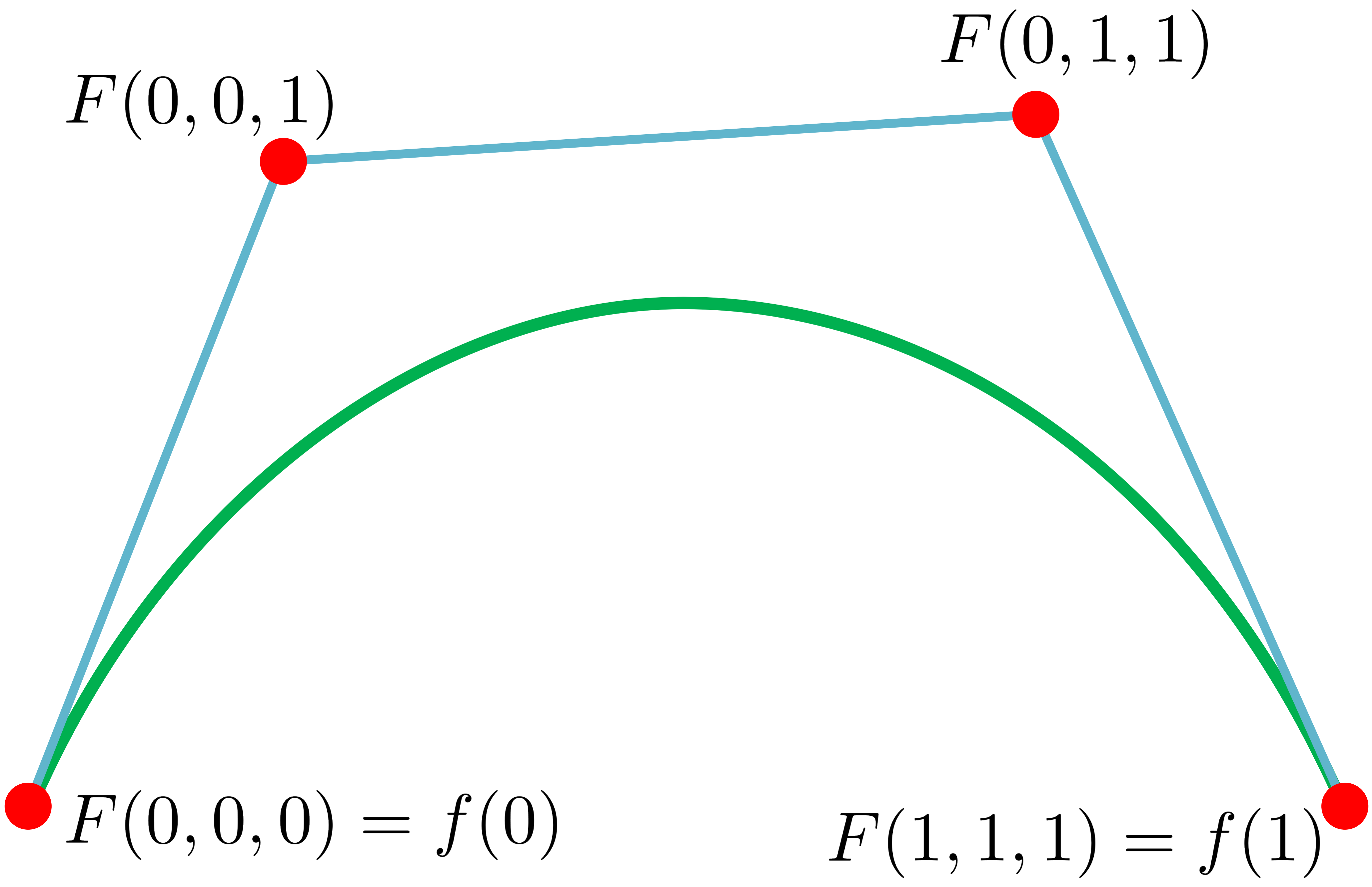
Instructor: Hao Su

Credit: Justin Solomon

Q:

What do these calculations look like in software?

# Old-School Approach




Piecewise smooth approximations

## Question

# What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\| \, dt$$

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What is the arc length of a cubic Bézier curve?

$$\int_a^b \|\gamma'(t)\| \, dt$$

Not known in closed form.

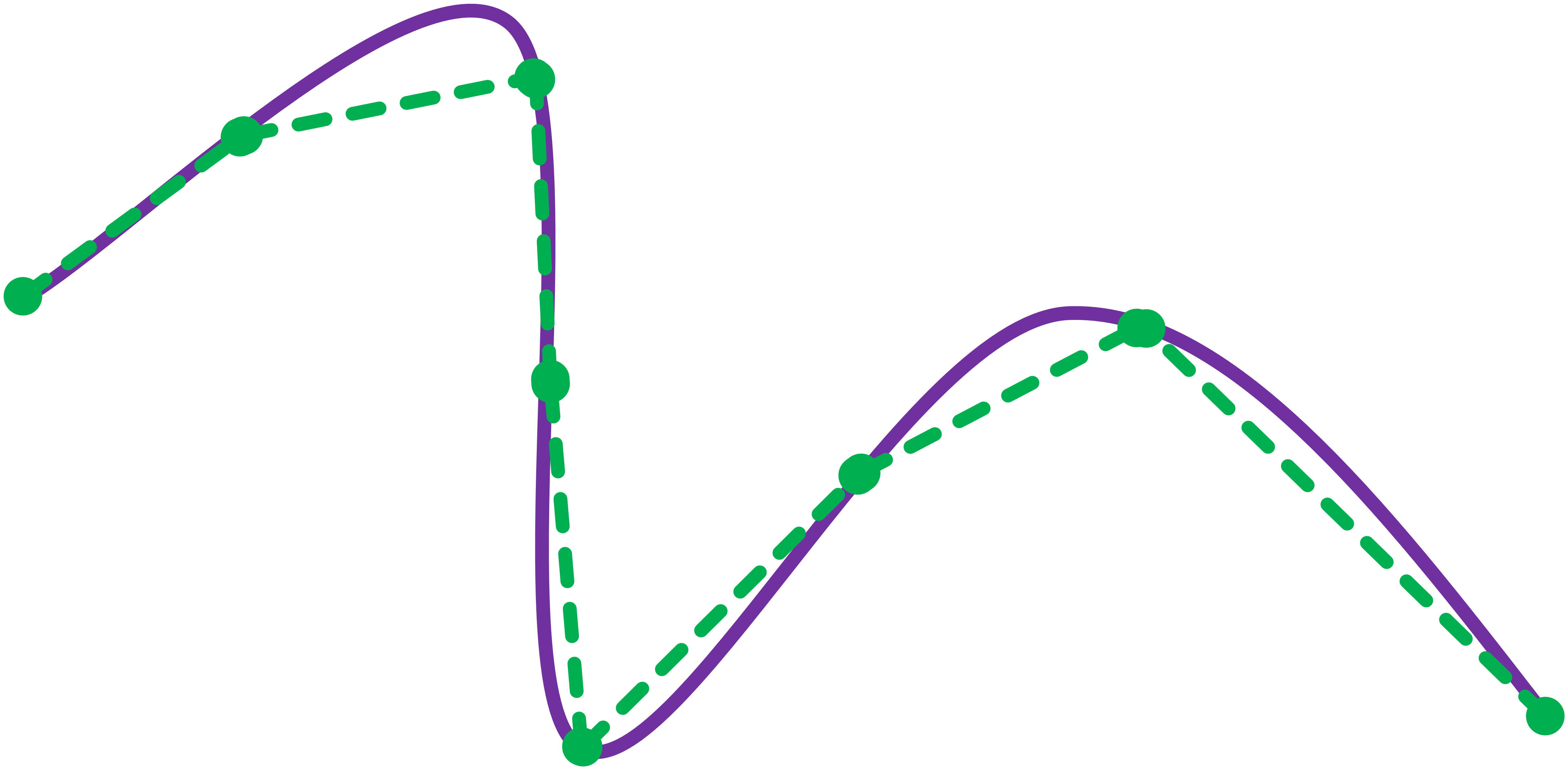
*Sad fact:*

Closed-form expressions rarely exist. When they do exist, they usually are messy.

# Only Approximations Anyway

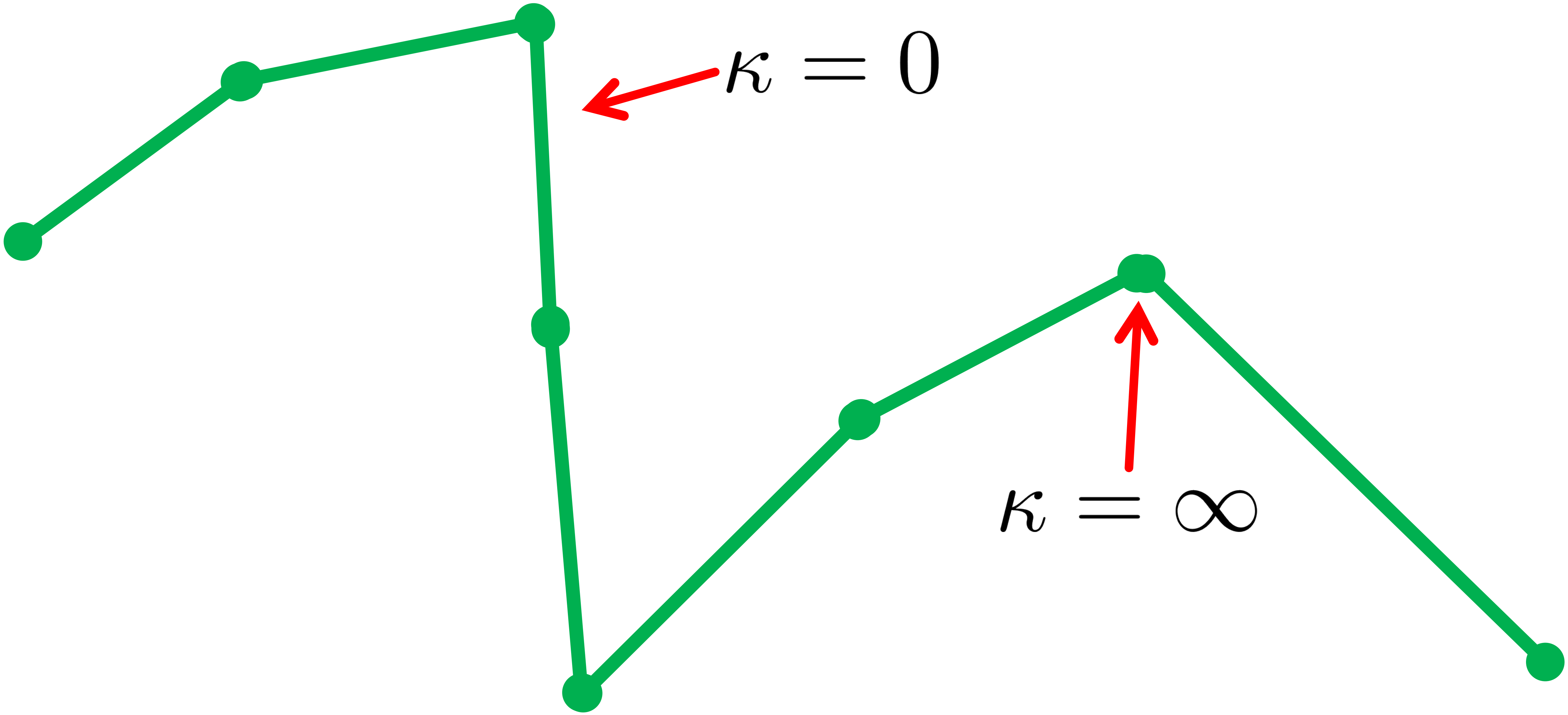
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \to \mathbb{R}^3\}$$

# Equally Reasonable Approximation



Piecewise linear

# Big Problem




Boring differential structure

# Finite Difference Approach

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

THEOREM: As , [insert statement].

# Reality Check

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

THEO … ment].

$$h > 0$$

# Two Key Considerations

- Convergence to continuous theory
- Discrete behavior

# Goal

Examine discrete theories of differentiable curves.

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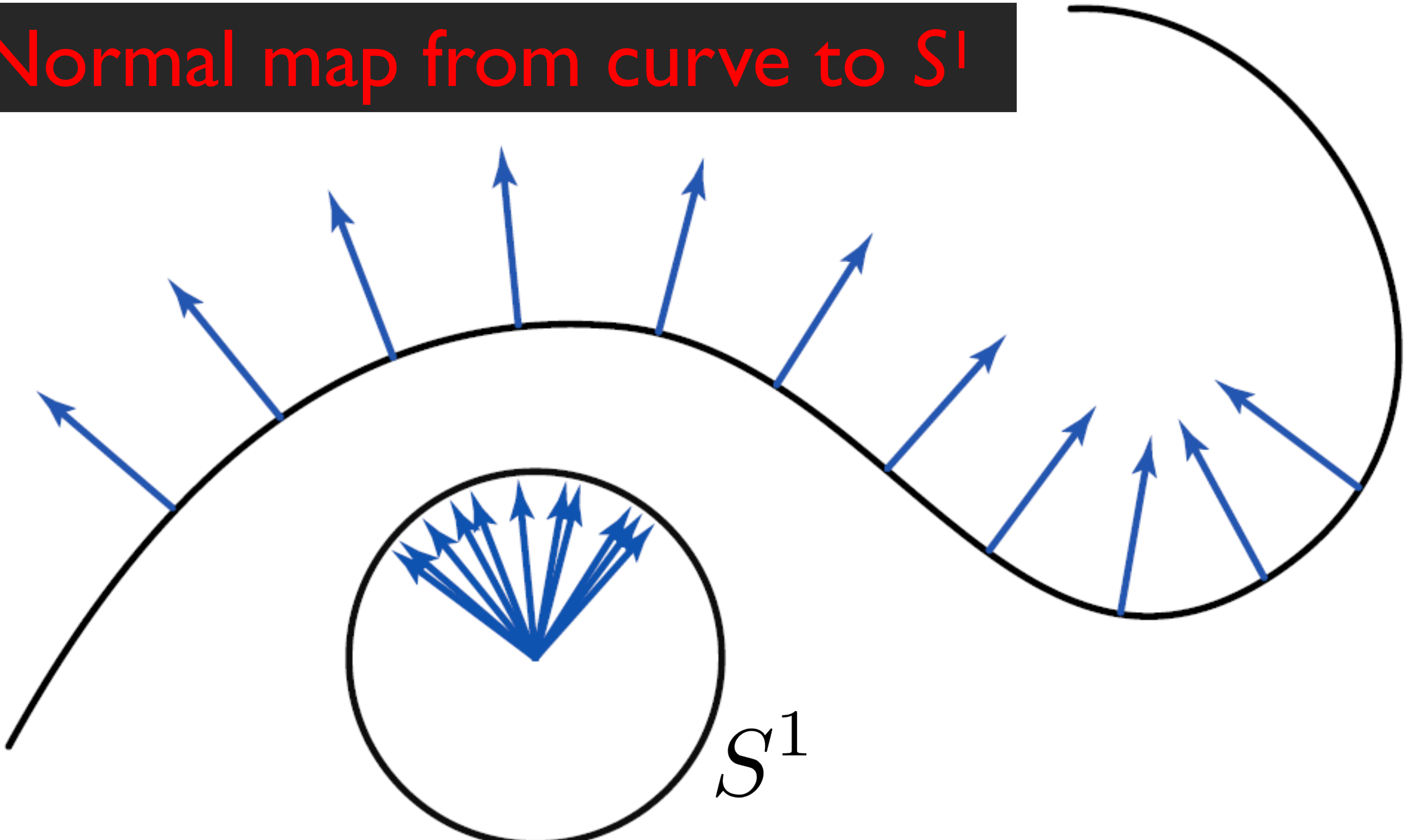
# Gauss Map

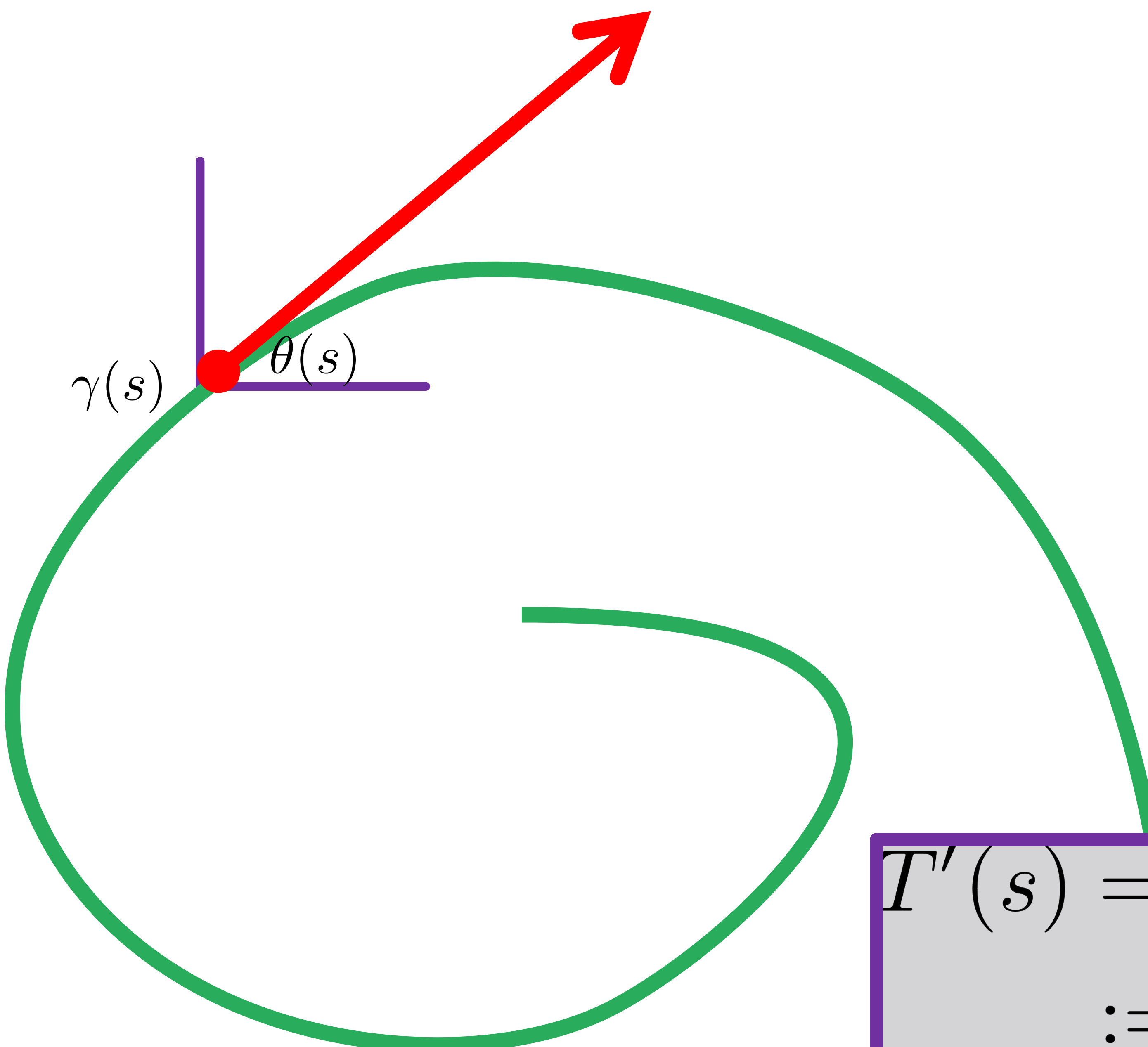
Normal map from curve to $S^1$




http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

# Signed Curvature on Plane Curves

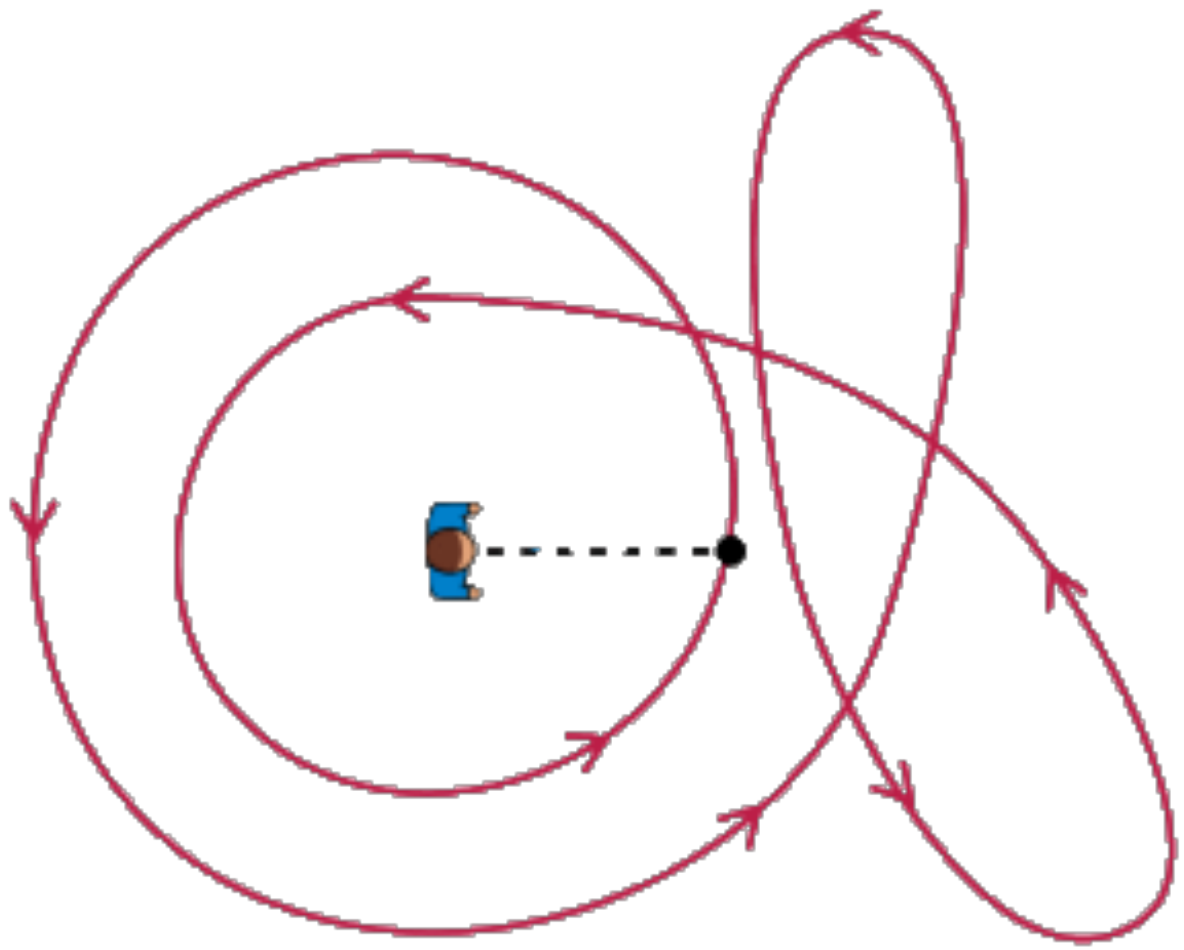
$$T(s) = (\cos\theta(s), \sin\theta(s))$$



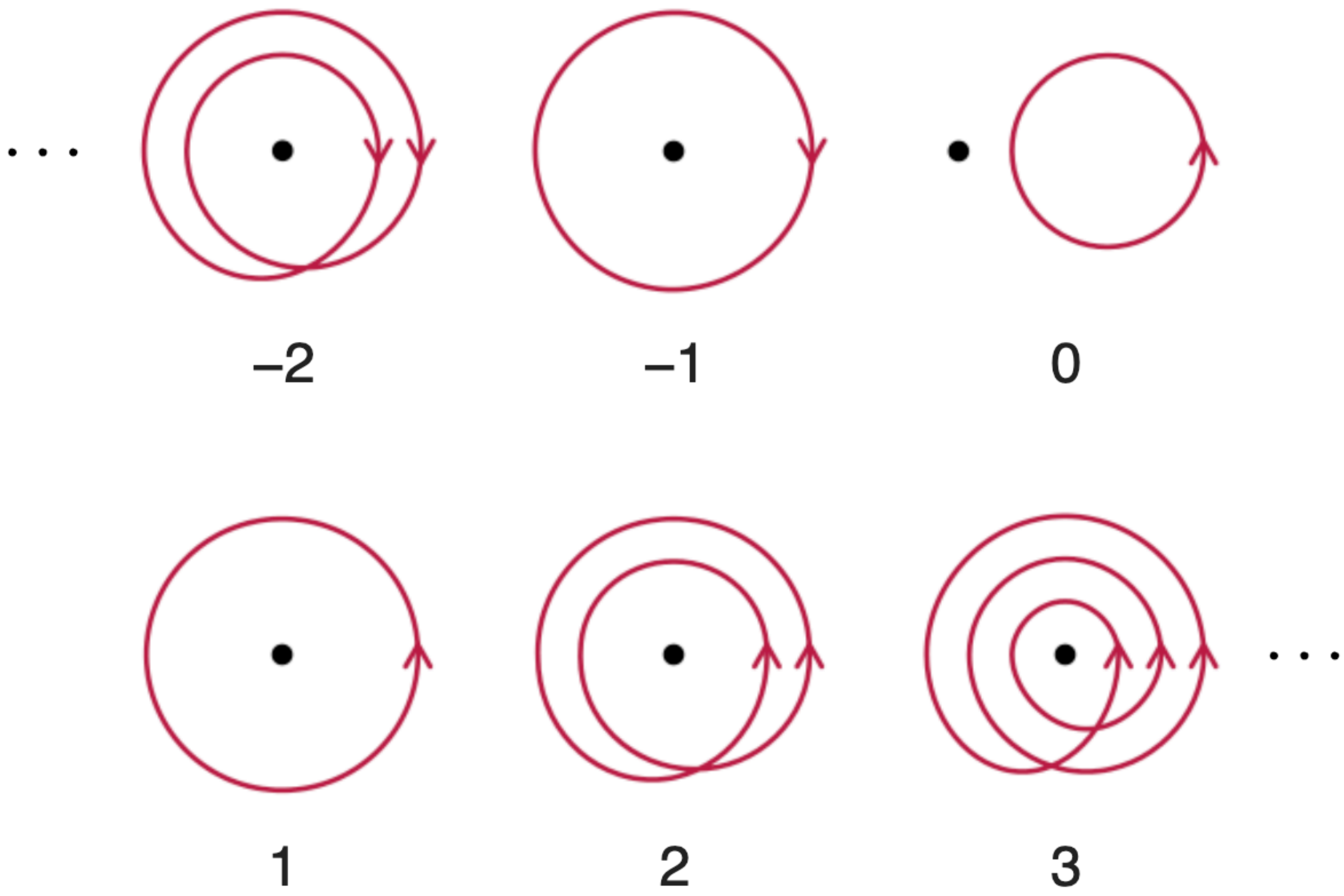

$$T'(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s))$$
$$:= \kappa(s)N(s)$$

# Winding Number

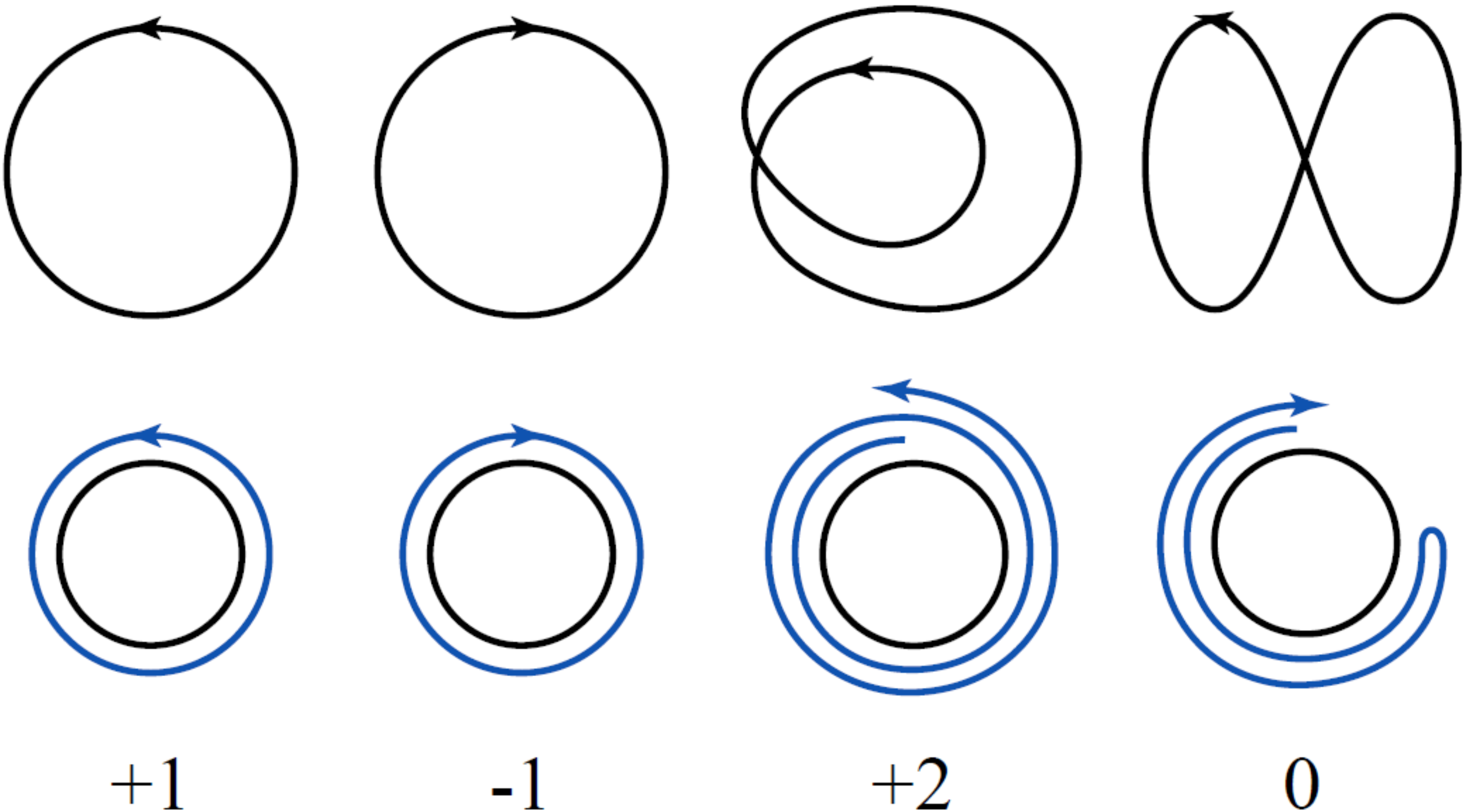
- The total number of times that curve travels counterclockwise around the point.
- The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise.



# Winding Number



# Turning Numbers




http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

# Recovering Theta

$$\theta'(s) = \kappa(s)$$

$$\Downarrow$$

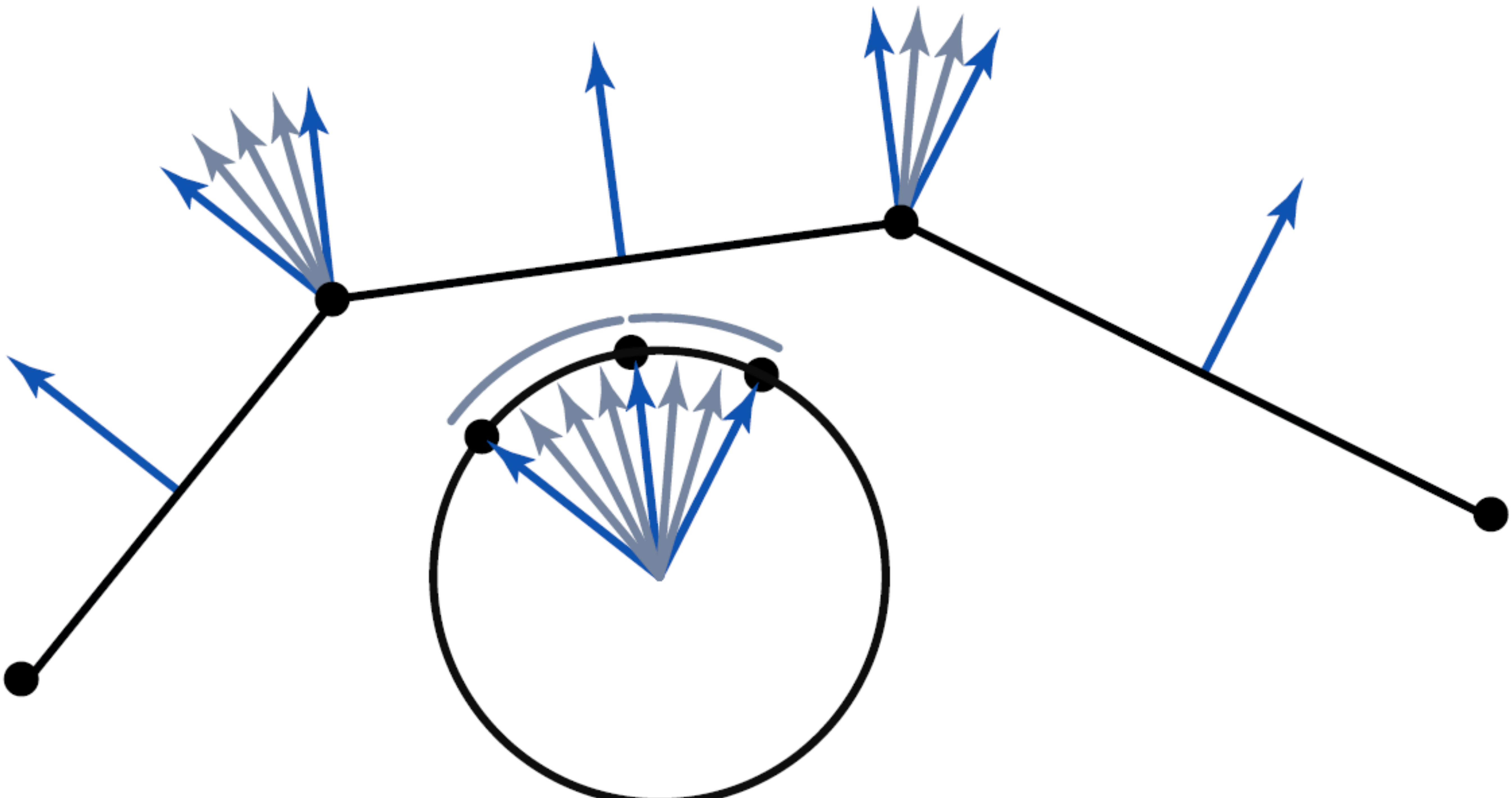
$$\Delta\theta = \int_{s_0}^{s_1} \kappa(s)\, ds$$

# Turning Number Theorem

$$\int_\Omega \kappa(s)\, ds = 2\pi k$$

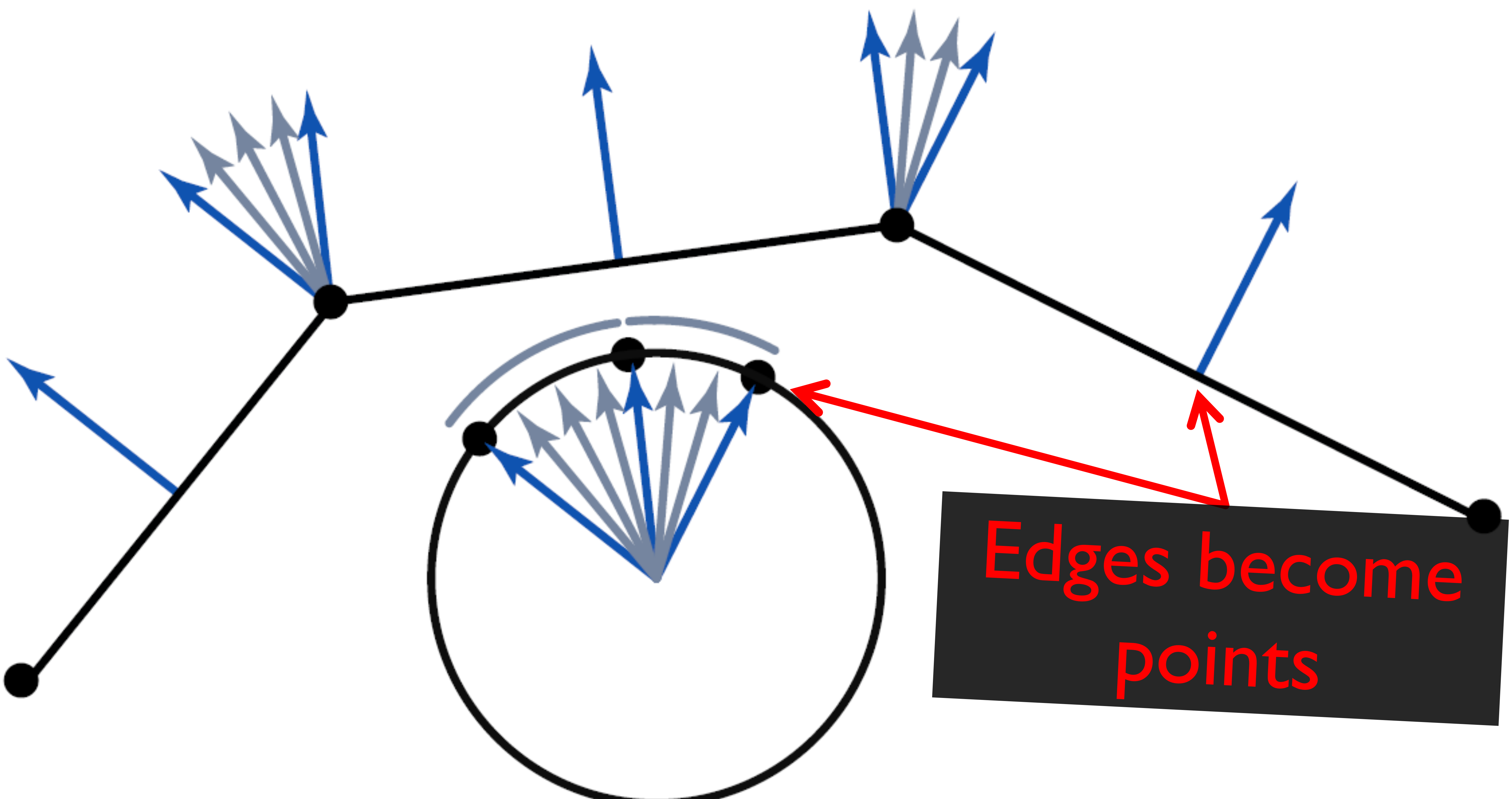
A “global” theorem!

# Discrete Gauss Map



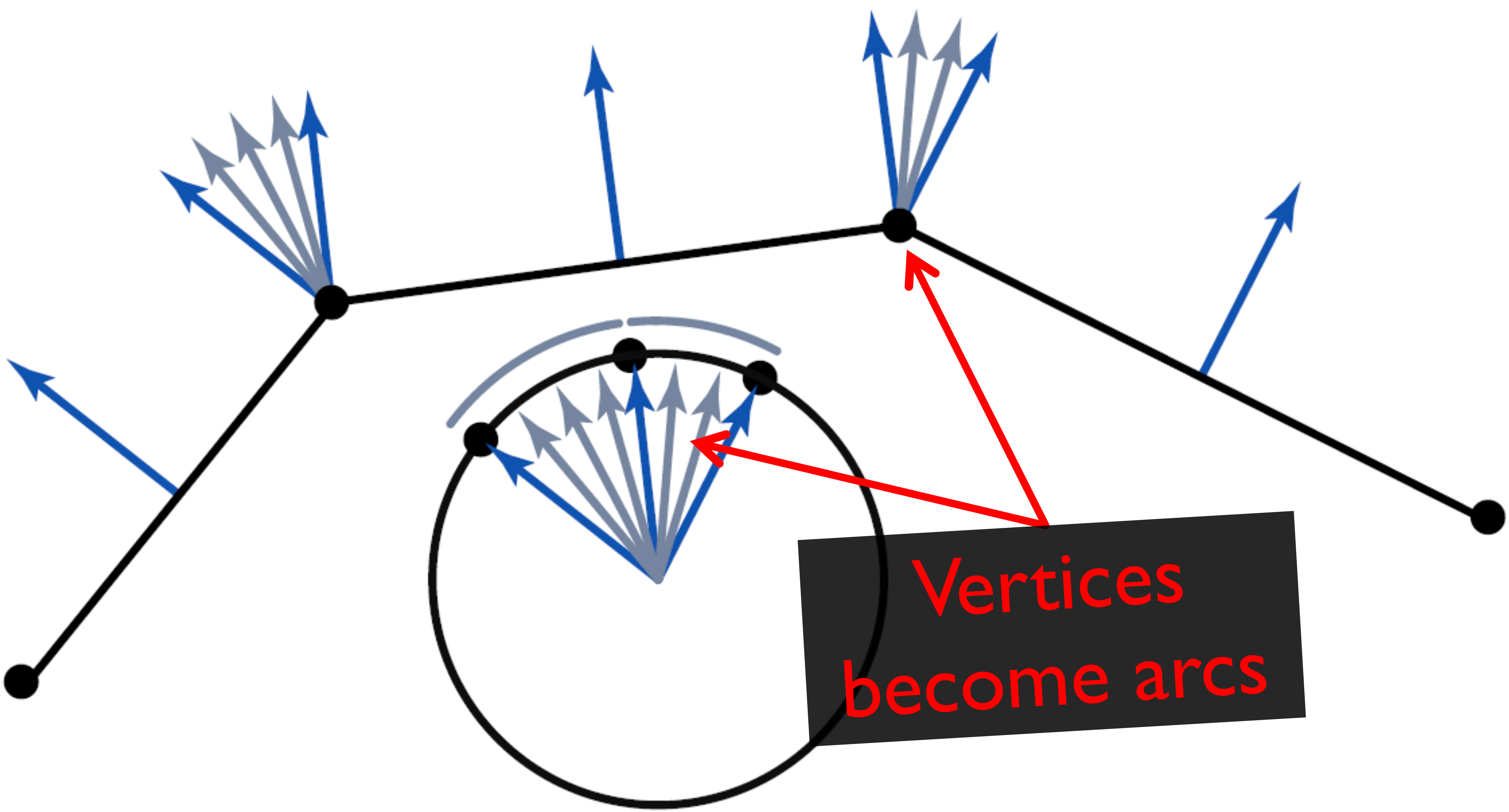
http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

# Discrete Gauss Map




http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

# Discrete Gauss Map

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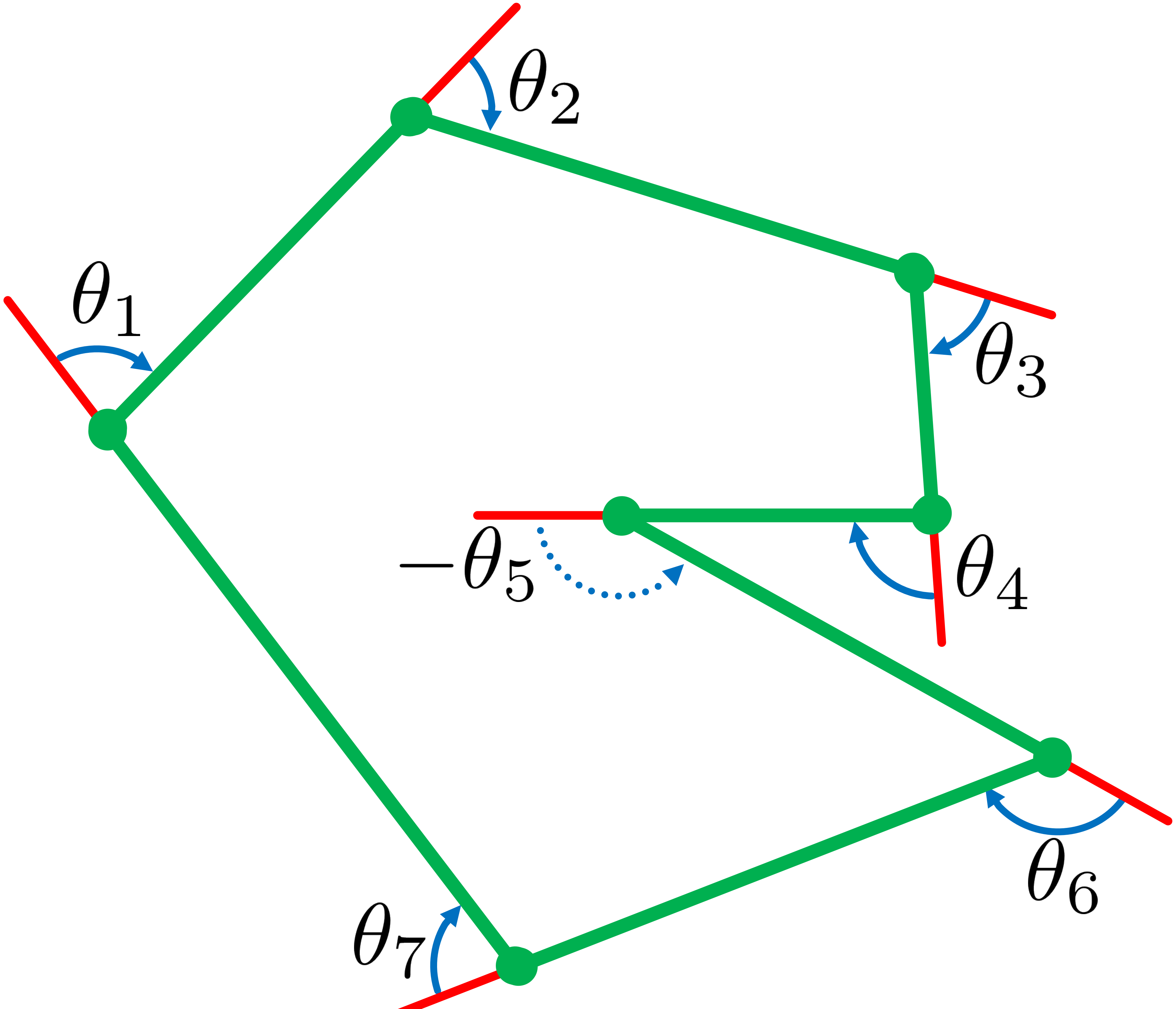
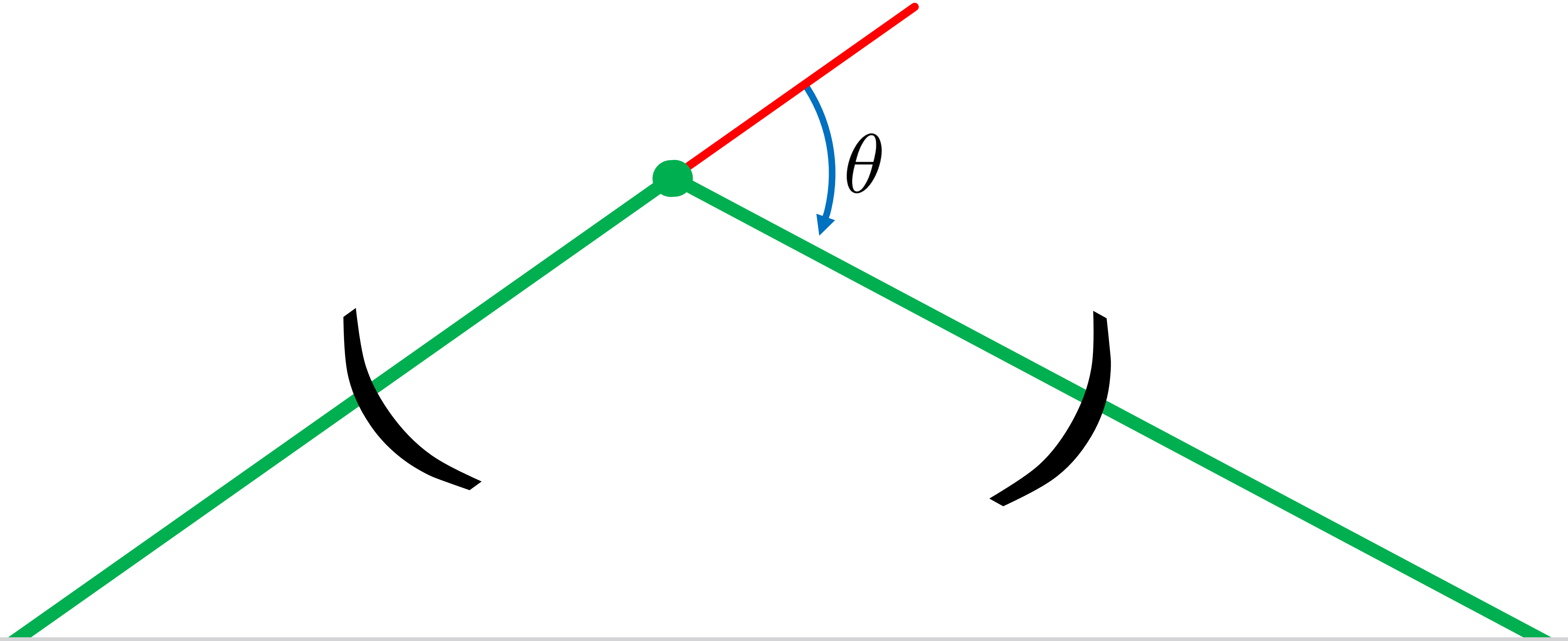
# Key Observation




$$\sum_i \theta_i = 2\pi k$$

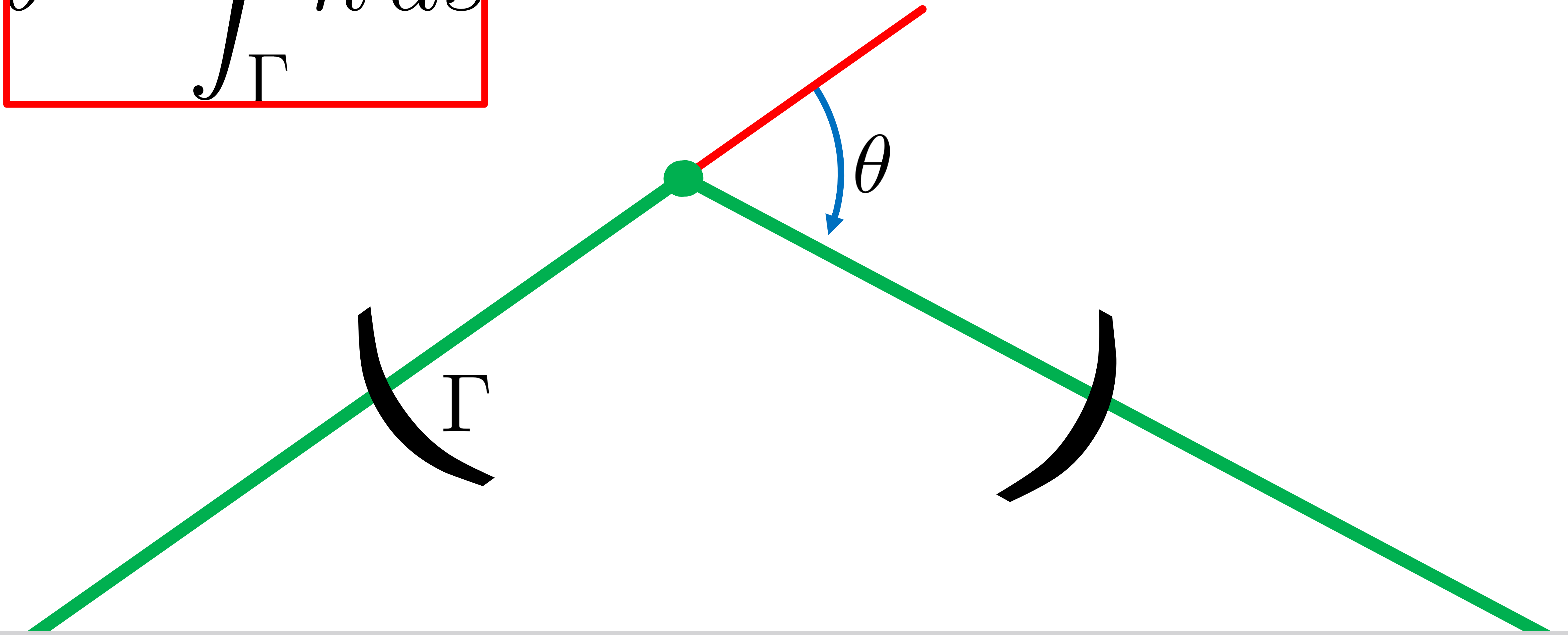
# What's Going On?




Total change in curvature

# What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

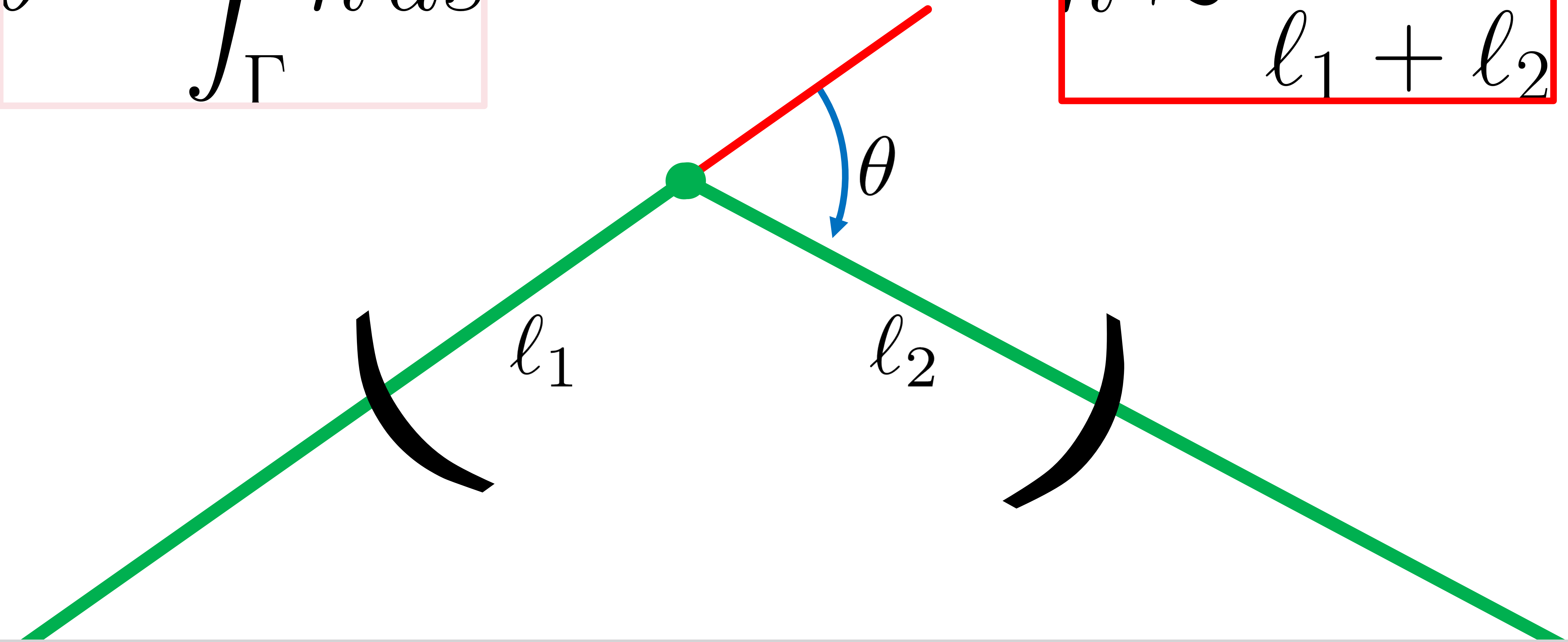



Total change in curvature

# What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

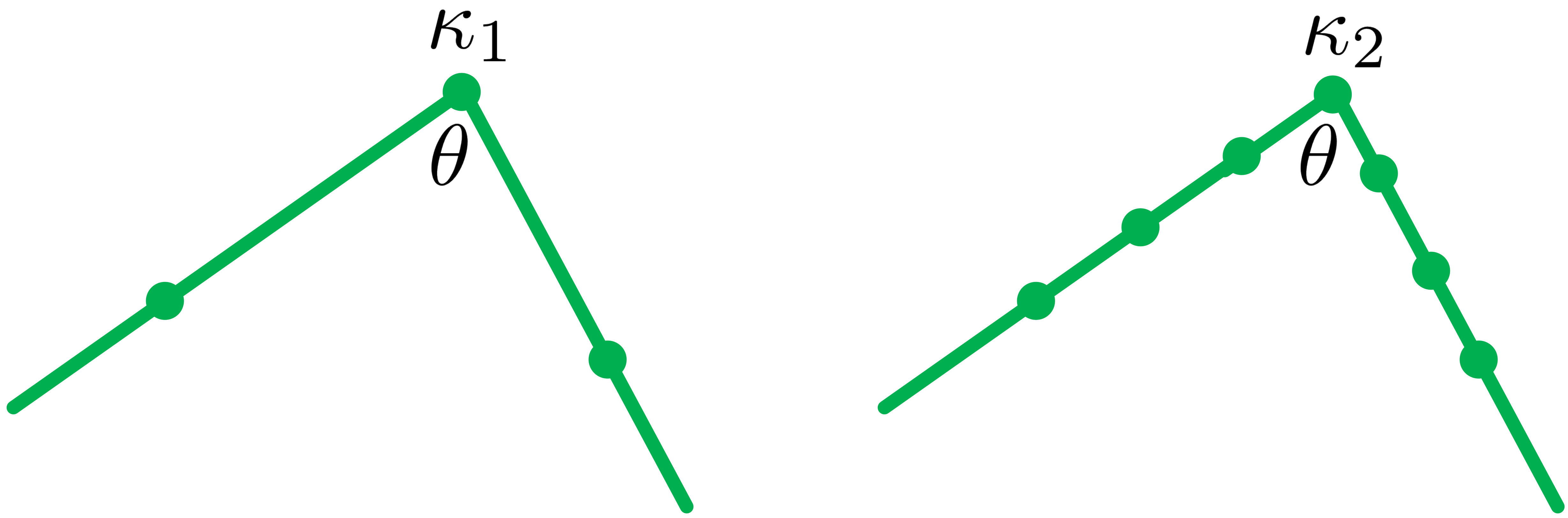
$$\kappa \approx \frac{\theta}{\ell_1 + \ell_2}$$




Total change in curvature

# Interesting Distinction

$$\kappa_1 \neq \kappa_2$$




Same integrated curvature

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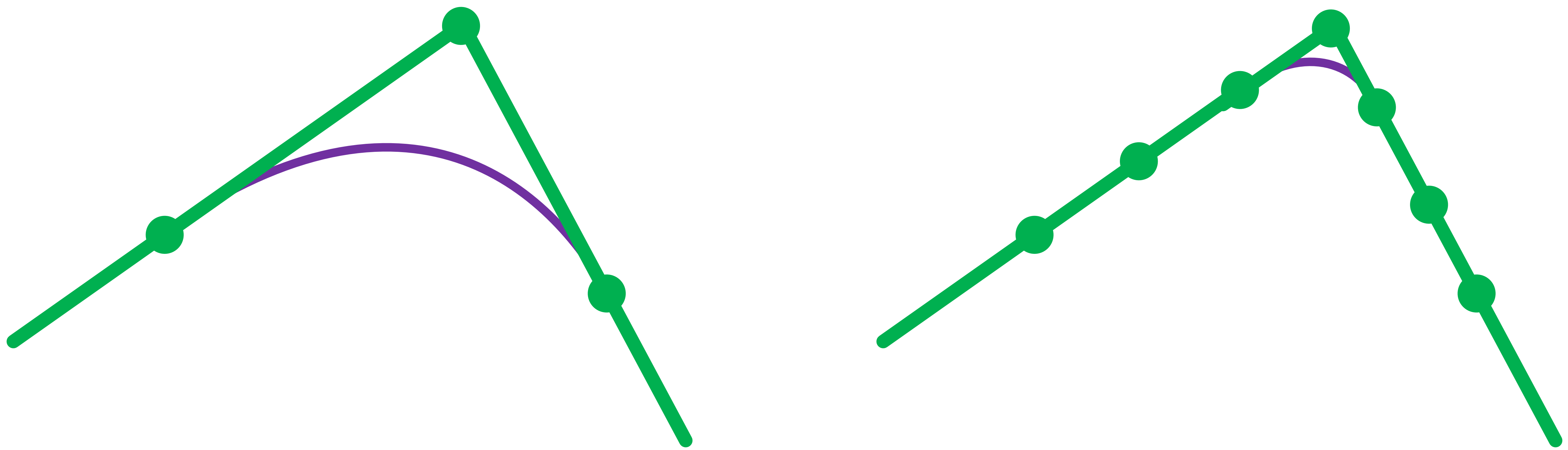

Same integrated curvature

# What's Going On?

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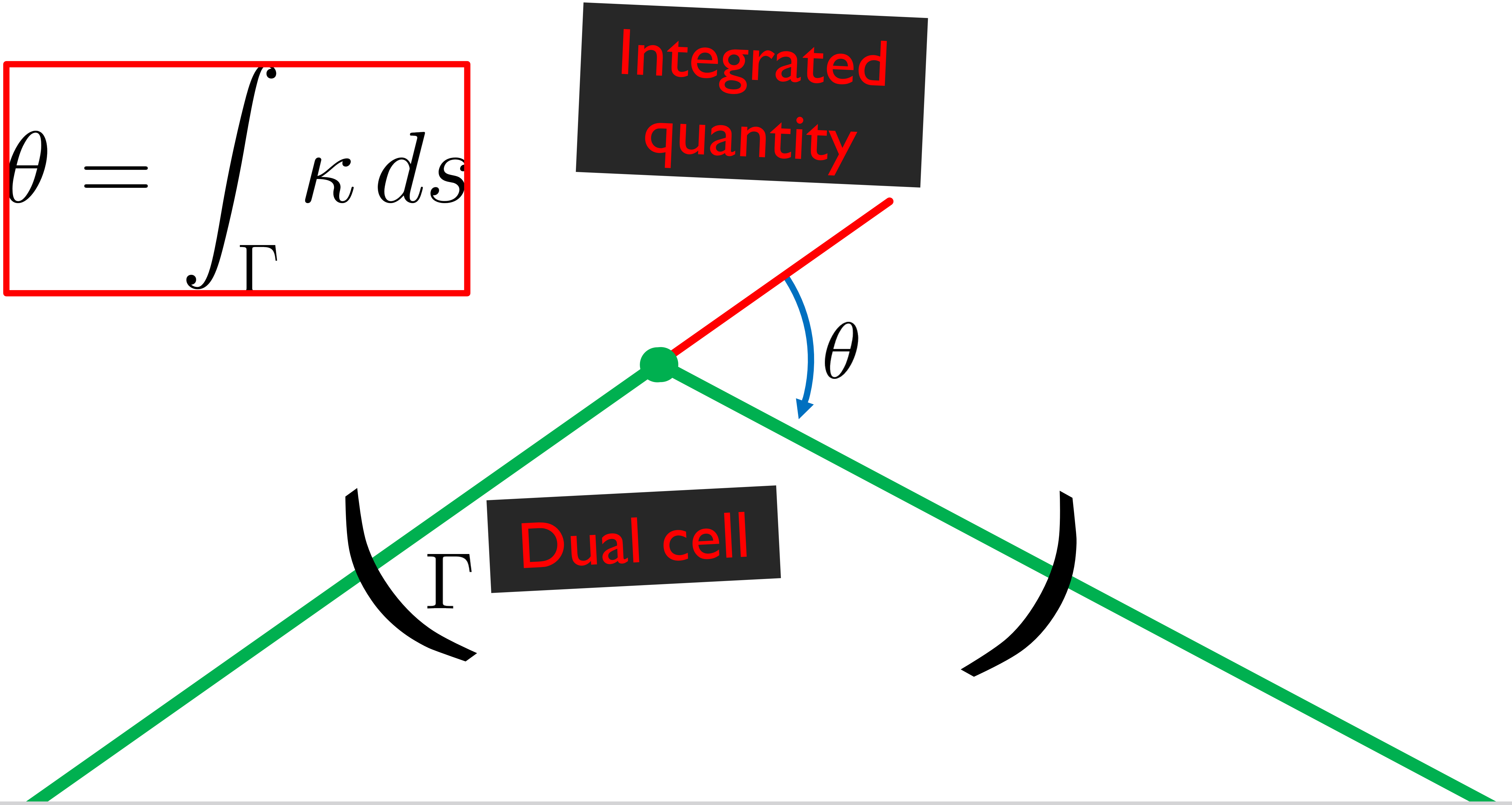
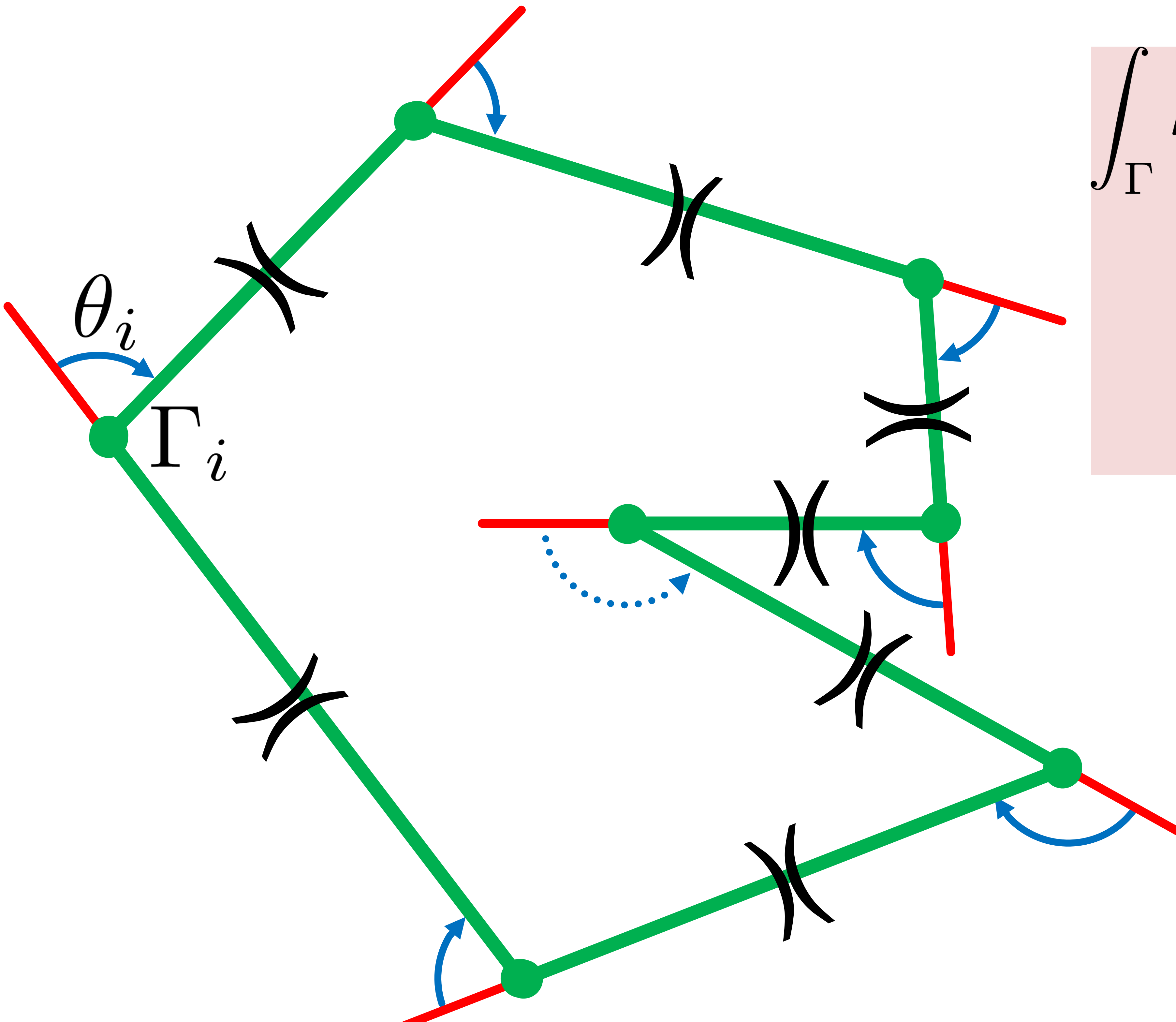



Total change in curvature

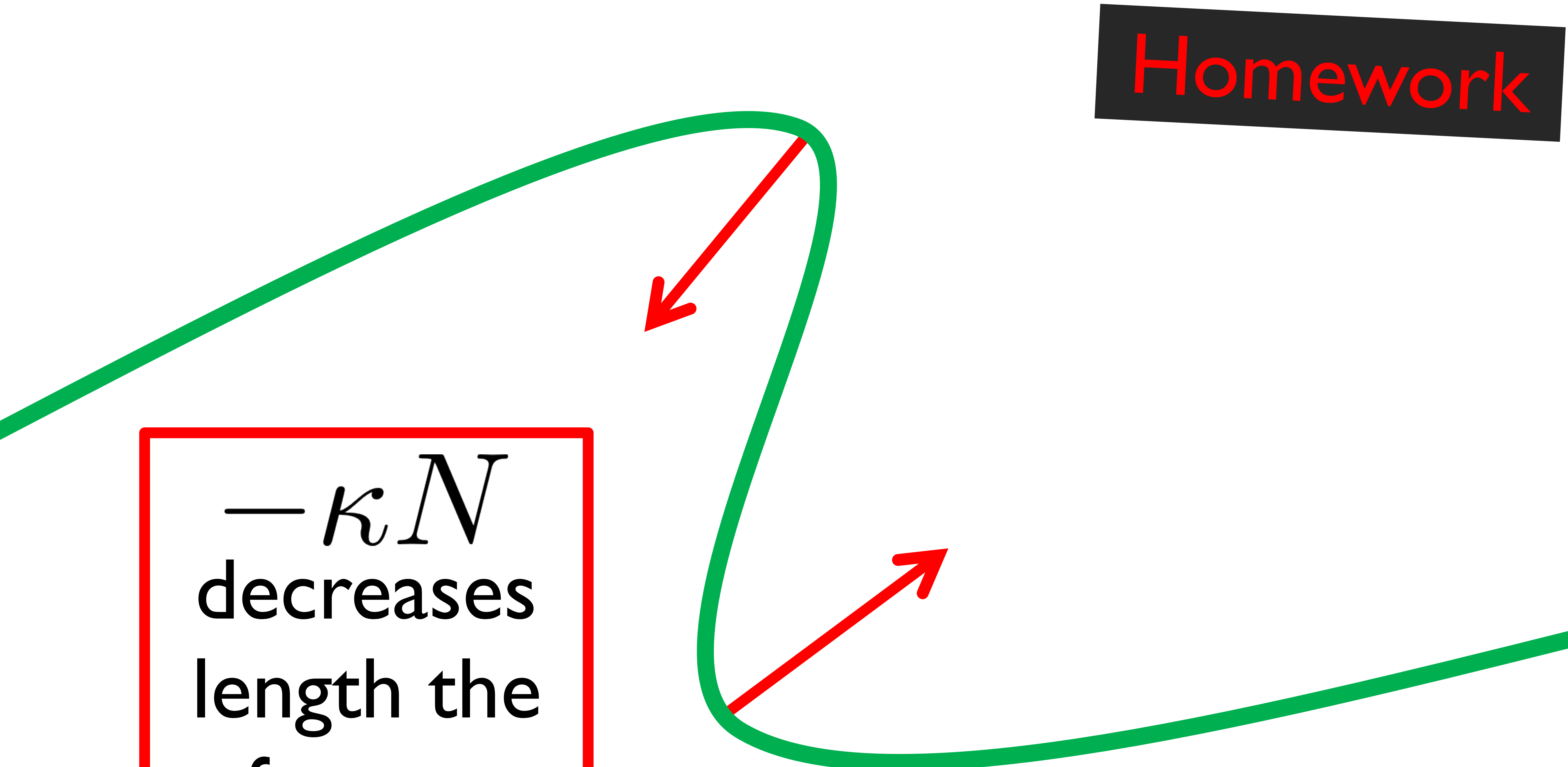
# Discrete Turning Angle Theorem




$$\int_\Gamma \kappa \, ds = \sum_i \int_{\Gamma_i} \kappa \, ds$$
$$= \sum_i \theta_i$$
$$= 2\pi k$$

Preserved structure!

# Alternative Definition




$-\kappa N$ decreases length the fastest.

# Remaining Question

Does discrete curvature converge in limit?

Yes!

# Remaining Question

Questions:

- Type of convergence?
- Sampling?
- Class of curves?

**Does discrete curvature converge in limit?**

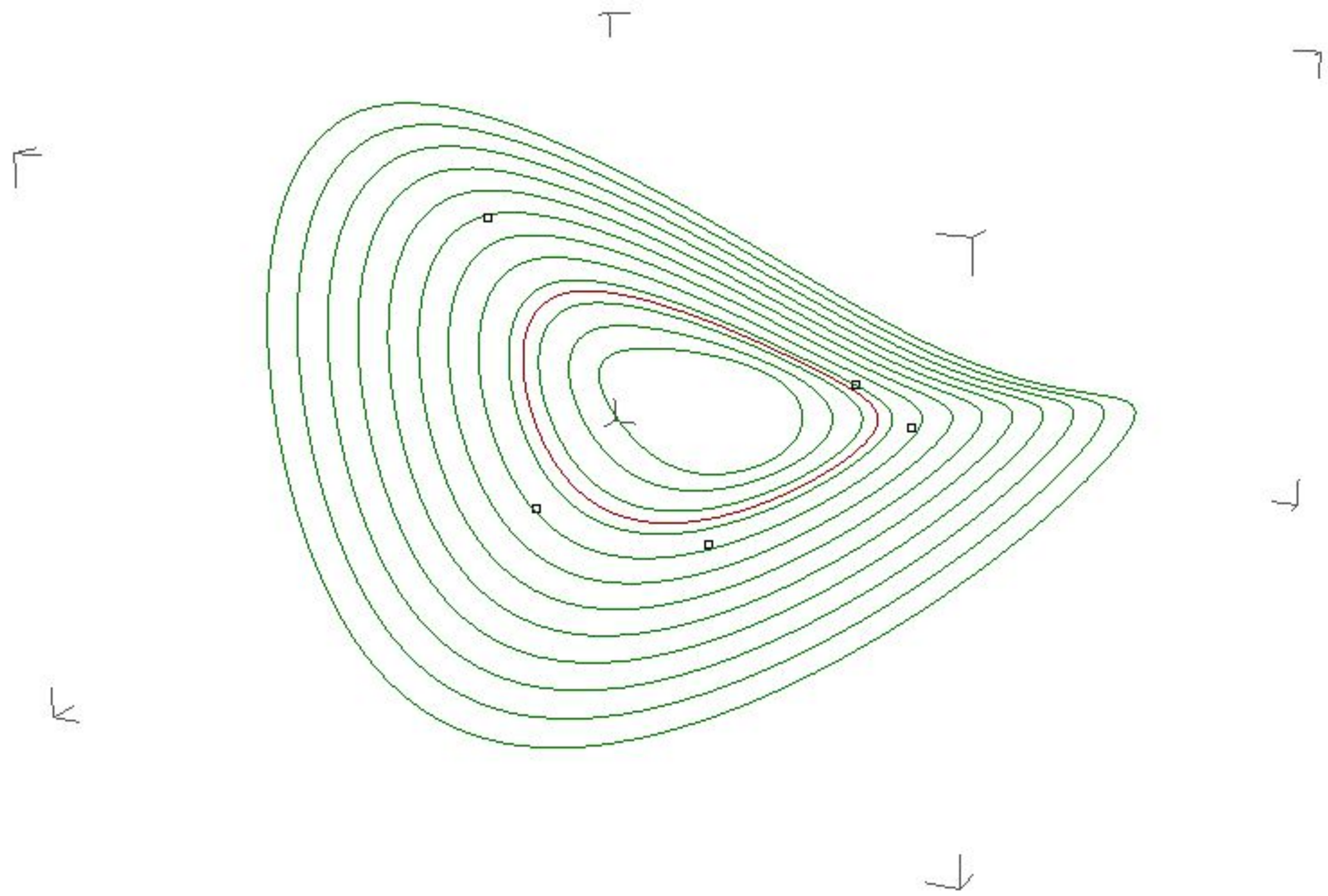
Yes!

# Discrete Differential Geometry

- Different discrete behavior

- Same convergence

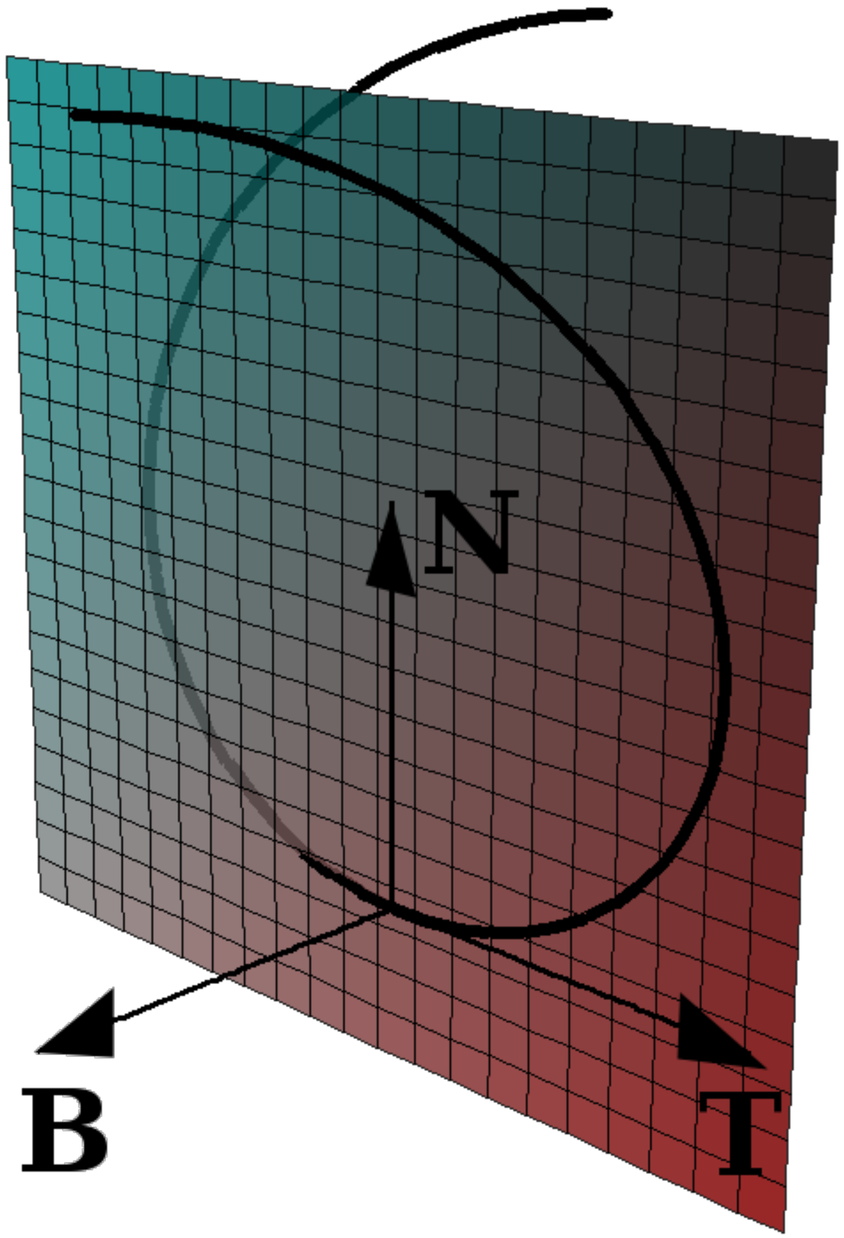
# Next



http://www.grasshopper3d.com/forum/topics/offseting-3d-curves-component

# Curves in 3D

# Frenet Frame




$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

# Potential Discretization

$$T_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|}$$

$$B_j = t_{j-1} \times t_j$$

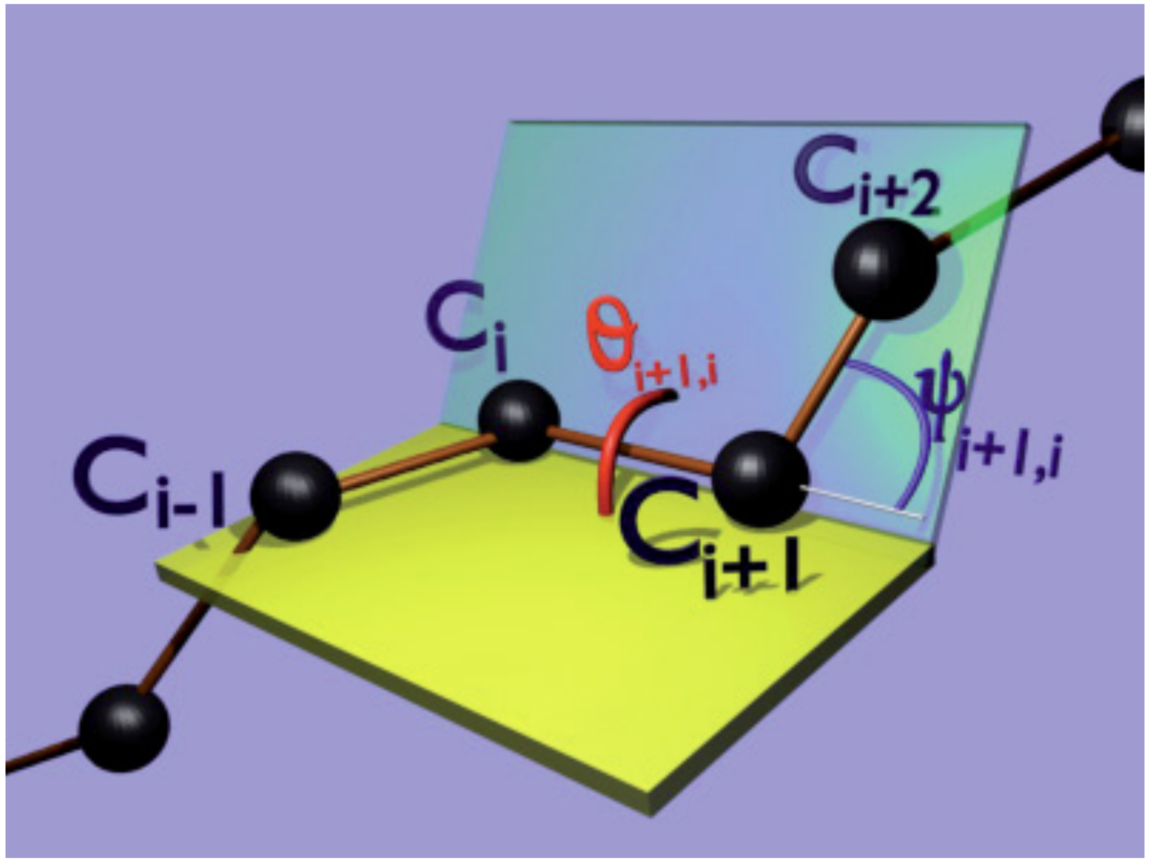
$$N_j = b_j \times t_j$$

Discrete Frenet frame

Discrete frame introduced in:
**The resultant electric moment of complex molecules**
Eyring, Physical Review, 39(4):746—748, 1932.

# Transfer Matrix

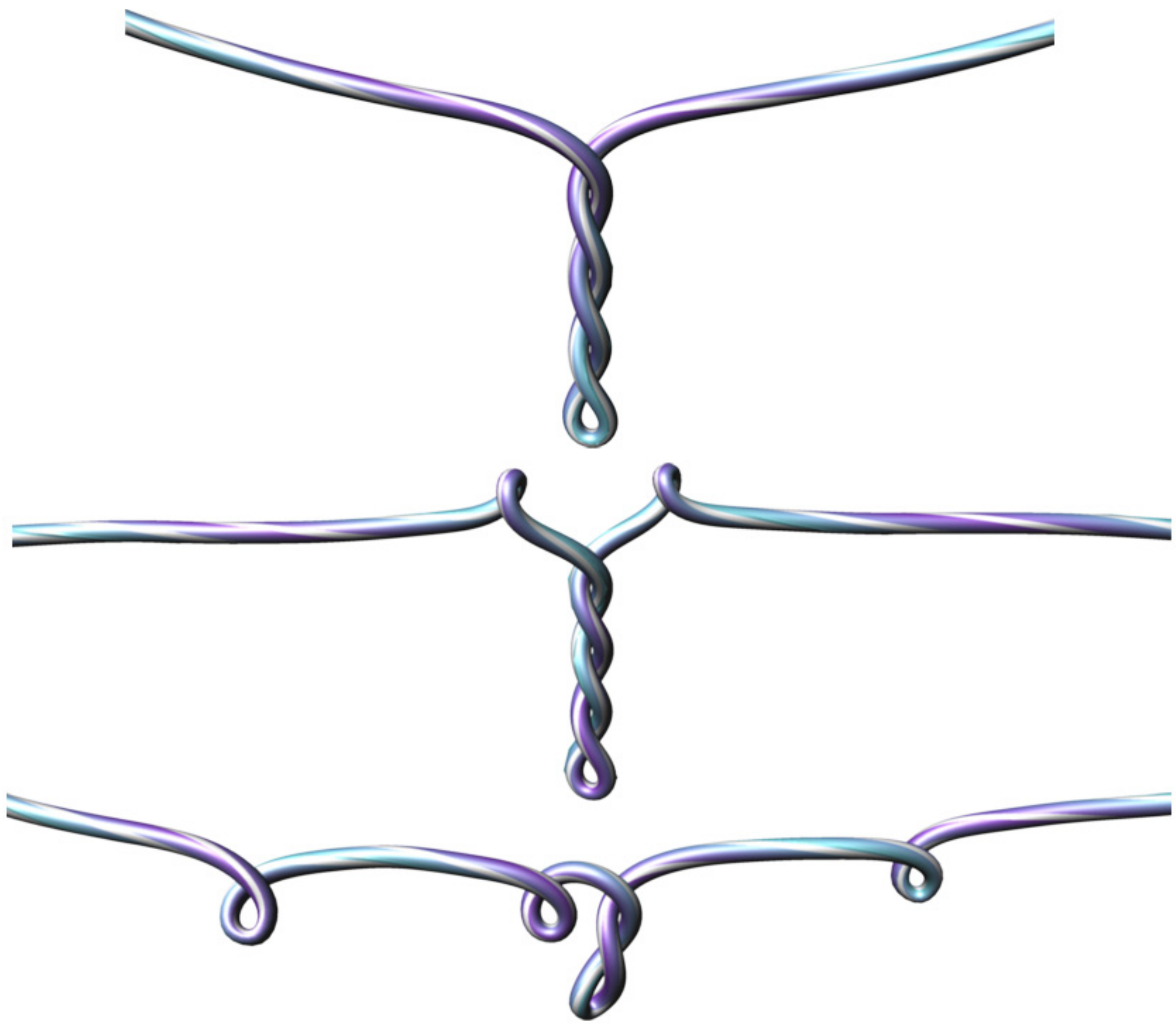
$$\begin{pmatrix} T_{i+1} \\ N_{i+1} \\ B_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} T_i \\ N_i \\ B_i \end{pmatrix}$$




Discrete construction that works for fractal curves and converges in continuum limit.

Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins
Hu, Lundgren, and Niemi
*Physical Review E* 83 (2011)

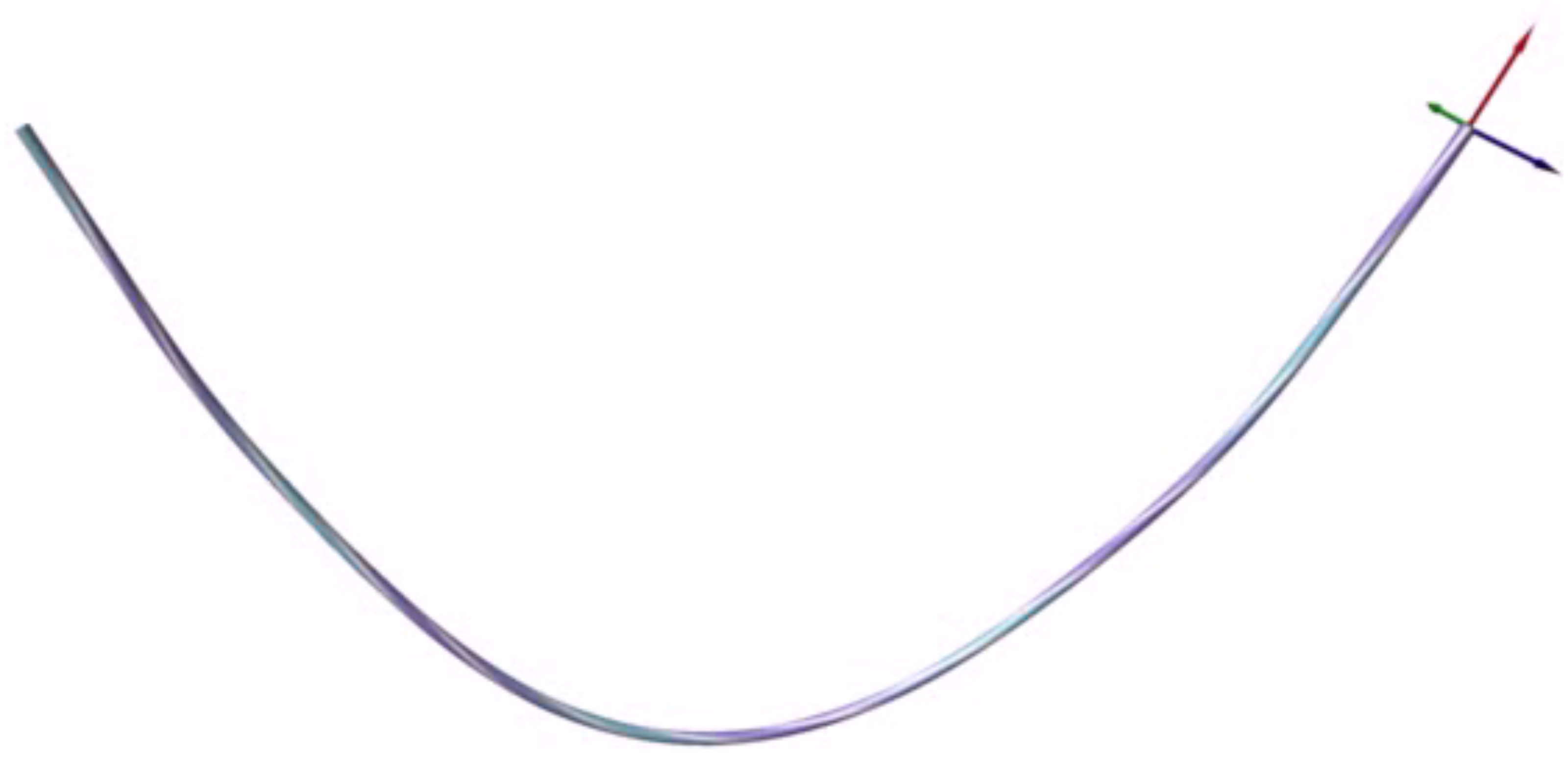
# Segments Not Always Enough



**Discrete Elastic Rods**
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
*SIGGRAPH* 2008

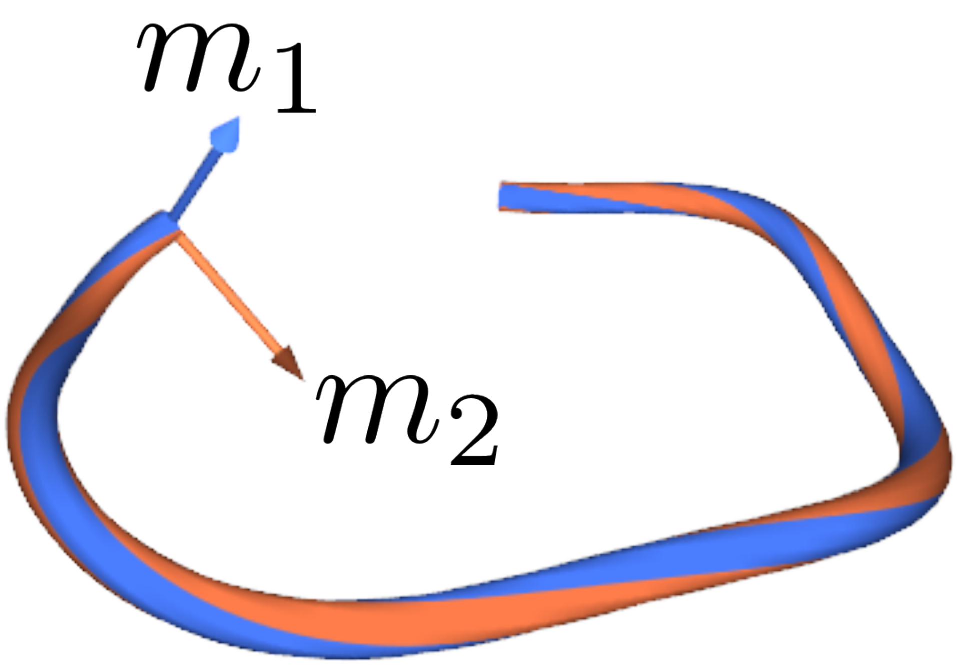
http://www.cs.columbia.edu/cg/rods/

# Simulation Goal



http://www.cs.columbia.edu/cg/rods/

# Adapted Framed Curve




$$\Gamma = \{\gamma(s); T, m_1, m_2\}$$

Material frame

http://www.cs.columbia.edu/cg/rods/

Normal part encodes twist

# Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{1}{2}\int_\Gamma \alpha\kappa^2\, ds$$

Punish turning the steering wheel

$$\begin{aligned}
\kappa N &= T' \\
&= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\
&= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\
&:= \omega_1 m_1 + \omega_2 m_2
\end{aligned}$$

# Bending Energy

$$E_{\mathrm{bend}}(\Gamma) := \frac{1}{2}\int_\Gamma \alpha(\omega_1^2 + \omega_2^2)\,ds$$

Punish turning the steering wheel

$$\begin{aligned}\kappa N &= T' \\ &= (T' \cdot T)T + (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &= (T' \cdot m_1)m_1 + (T' \cdot m_2)m_2 \\ &:= \omega_1 m_1 + \omega_2 m_2\end{aligned}$$

# Twisting Energy

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta m^2\, ds$$

Punish non-tangent change in material frame

$$\begin{aligned} m &:= m_1' \cdot m_2 \\ &= \frac{d}{dt}(m_1 \cdot m_2) - m_1 \cdot m_2' \\ &= -m_1 \cdot m_2' \end{aligned}$$

# Twisting Energy

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta m^2 \, ds$$

Punish non-tangent change in material frame

$$m := m_1' \cdot m_2$$

$$= \frac{d}{dt}(m_1 \cdot m_2) - m_1 \cdot m_2'$$

$$= -m_1 \cdot m_2'$$

Swapping and does not affect !

# Which Basis to Use

THERE IS MORE THAN ONE WAY TO FRAME A CURVE

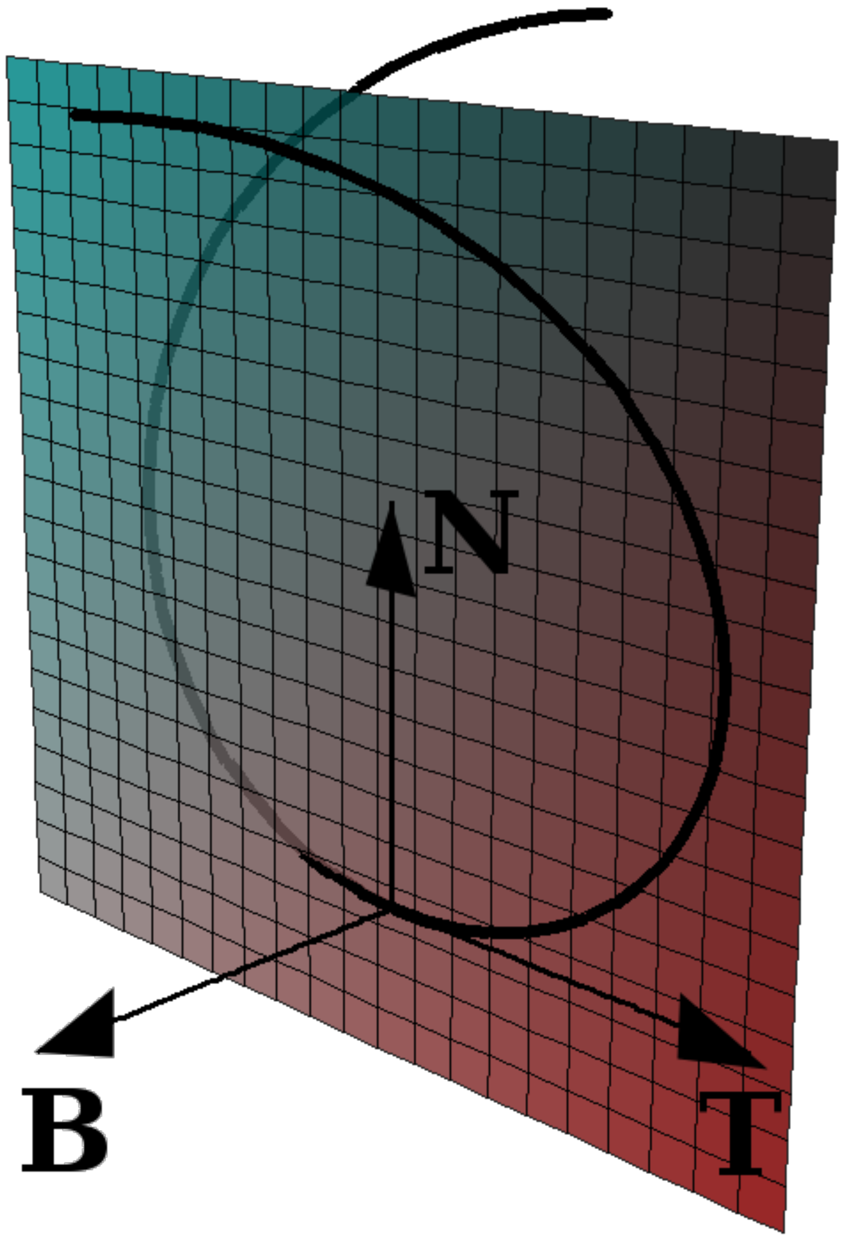
RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, $C^3$) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

**1. Relatively parallel fields.** We say that a normal vector field $M$ along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in

# Frenet Frame: Issue

?




$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

# Cross Product as Matrix Multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

$$[a_\times] = -[a_\times]^T$$

"skew-symmetric matrix"

# Darboux Vector of Frenet Frame

In terms of the Frenet-Serret apparatus, the Darboux vector **ω** can be expressed as[3]

$$\Omega = \tau \mathbf{T} + \kappa \mathbf{B} \qquad (1)$$

and it has the following symmetrical properties:[2]

$$\Omega \times \mathbf{T} = \mathbf{T}',$$

$$\Omega \times \mathbf{N} = \mathbf{N}',$$

$$\Omega \times \mathbf{B} = \mathbf{B}',$$

which can be derived from Equation (1) by means of the Frenet-Serret theorem (or vice versa).

# Bishop Frame and its Darboux Vector

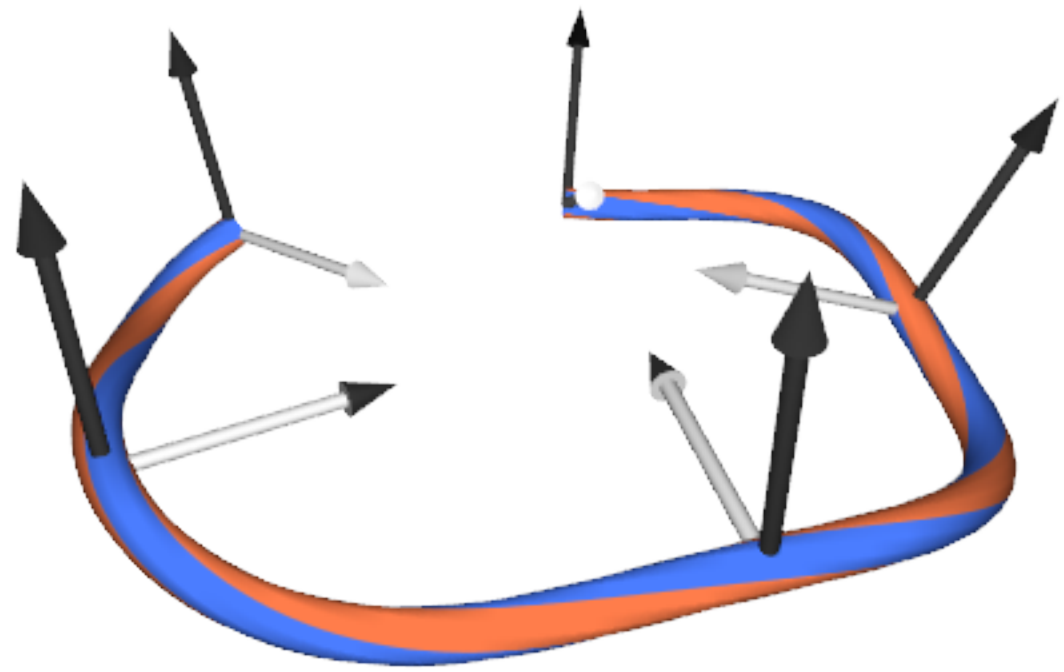
$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$\Omega := \kappa B \text{ (``curvature binormal'')}$$

Darboux vector



http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

# Bishop Frame

$$T' = \Omega \times T$$

$$u' = \Omega \times u$$

$$v' = \Omega \times v$$

$$\Omega := \kappa B \text{ (“curvature binormal”)}$$

$$u' \cdot v \equiv 0$$

No twist
(“parallel transport”)

http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

# Curve-Angle Representation

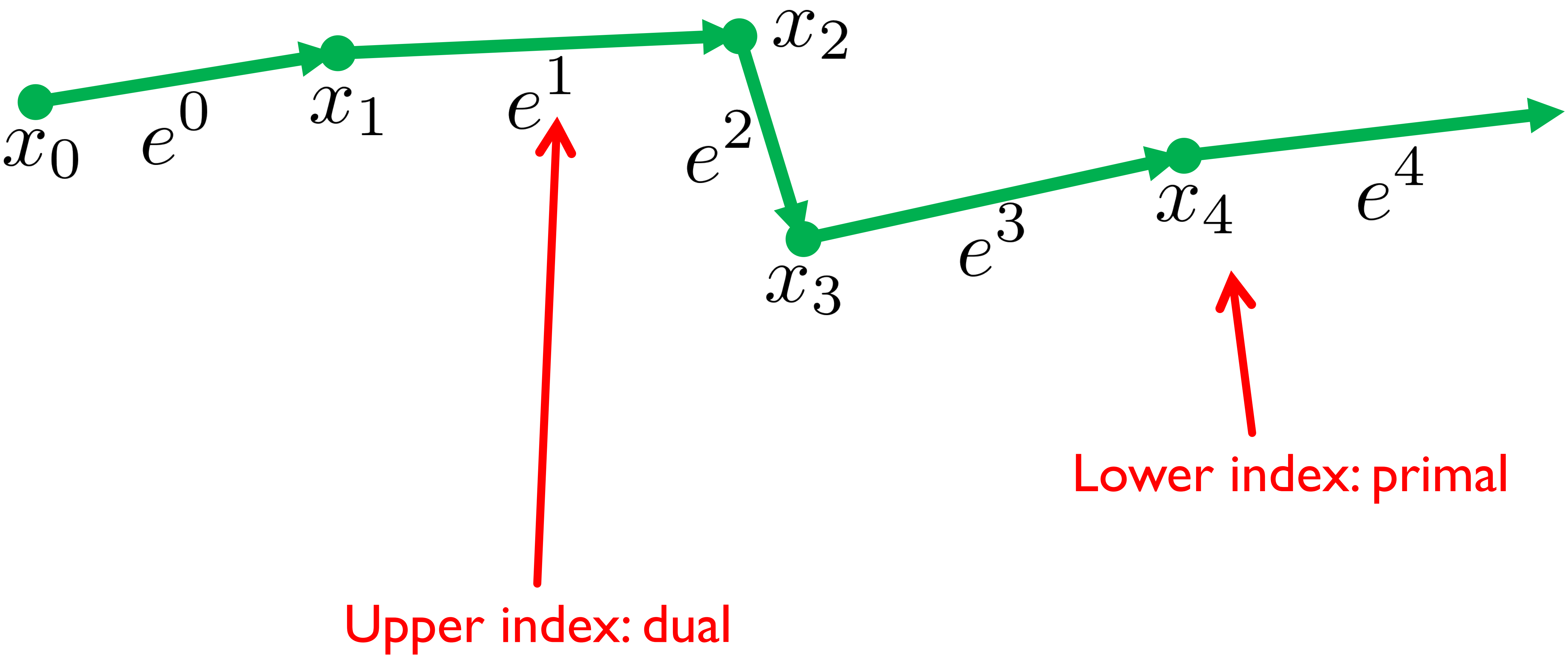
$$m_1 = u\cos\theta + v\sin\theta$$
$$m_2 = -u\sin\theta + v\cos\theta$$

$$E_{\text{twist}}(\Gamma) := \frac{1}{2}\int_\Gamma \beta(\theta')^2\,ds$$

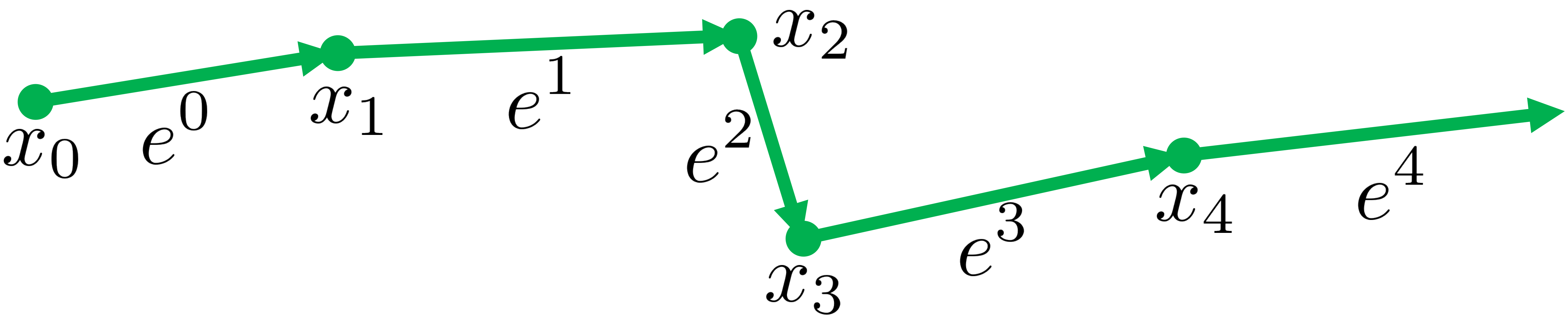
Degrees of freedom for elastic energy:

- Shape of curve
- Twist angle

# Discrete Kirchoff Rods



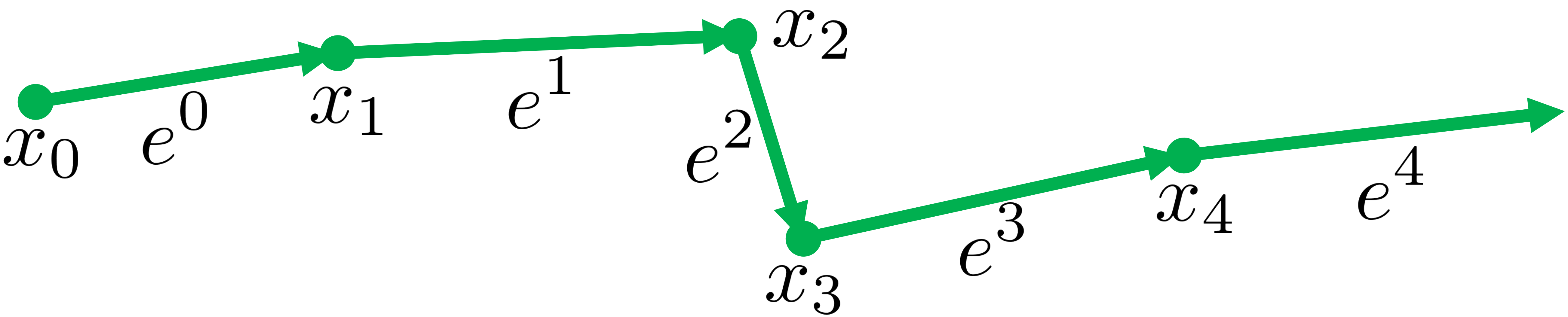
# Discrete Kirchoff Rods




$$T^i := \frac{e^i}{\|e^i\|}$$

Tangent unambiguous on edge

# Discrete Kirchoff Rods



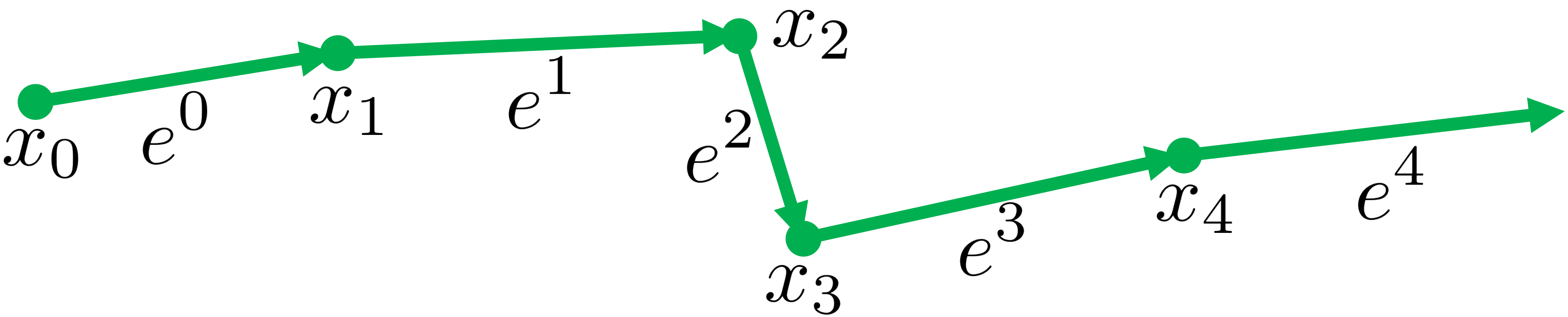

$$\kappa_i := 2\tan\frac{\phi_i}{2}$$

Turning angle

Yet another curvature!

Integrated curvature

# Discrete Kirchoff Rods




$$\kappa_i := 2\tan\frac{\phi_i}{2}$$

Yet another curvature!

$$(\kappa B)_i := \frac{2e^{i-1} \times e^i}{\|e^{i-1}\|\|e^i\| + e^{i-1} \cdot e^i}$$

Orthogonal to osculating plane, norm

## Darboux vector

# Bending Energy

$$E_{\text{bend}}(\Gamma) := \frac{\alpha}{2} \sum_i \left( \frac{(\kappa B)_i}{\ell_i / 2} \right)^2 \frac{\ell_i}{2}$$

$$= \alpha \sum_i \frac{\|(\kappa B)_i\|^2}{\ell_i}$$

Can extend for natural bend

Convert to pointwise and integrate

# Discrete Parallel Transport

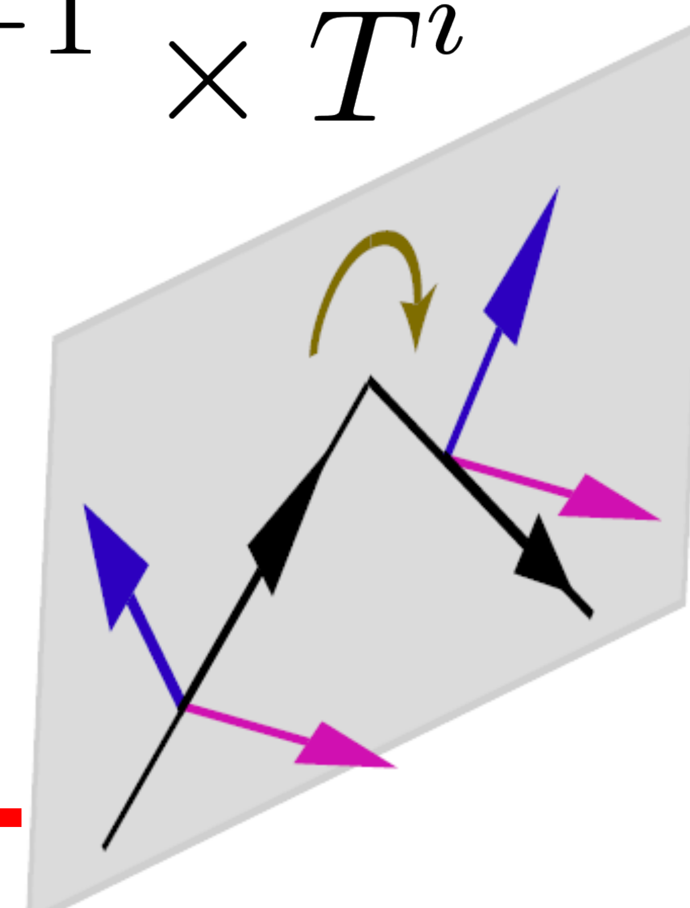
- Map tangent to tangent
- Preserve binormal
- Orthogonal

$$P_i(T^{i-1}) = T^i$$

$$P_i(T^{i-1} \times T^i) = T^{i-1} \times T^i$$

$$u^i = P_i(u^{i-1})$$

$$v^i = T^i \times u^i$$

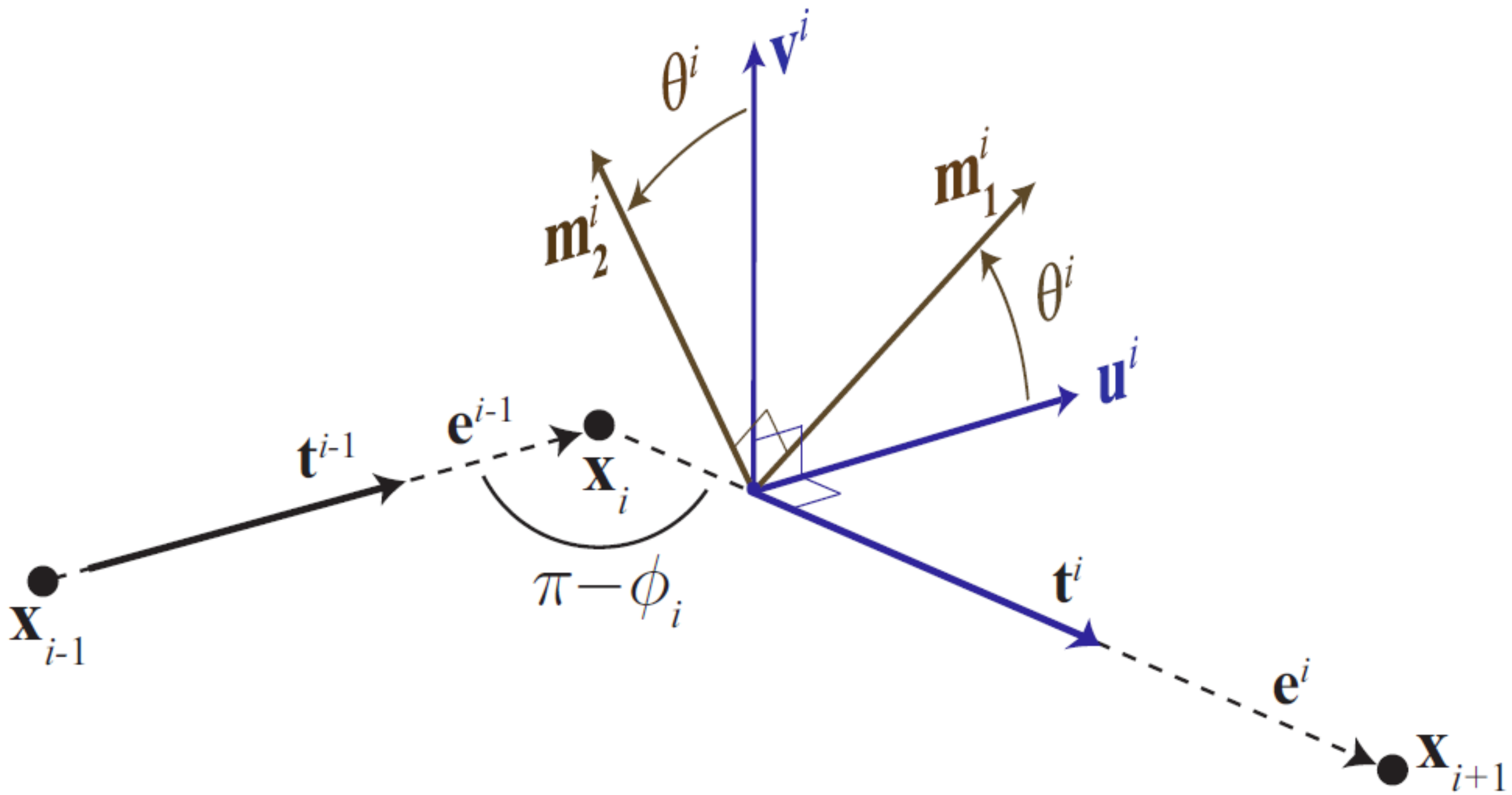


http://www.cs.columbia.edu/cg/rods/

# Discrete Material Frame

$$m_1^i = u^i \cos\theta^i + v^i \sin\theta^i$$

$$m_2^i = -u^i \sin\theta^i + v^i \cos\theta^i$$




http://www.cs.columbia.edu/cg/rods/

# Discrete Twisting Energy

$$E_{\text{twist}}(\Gamma) := \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$

Note can be arbitrary

# Simulation

`\omit{physics}`

Worth reading!

# Extension and Speedup

## Discrete Viscous Threads

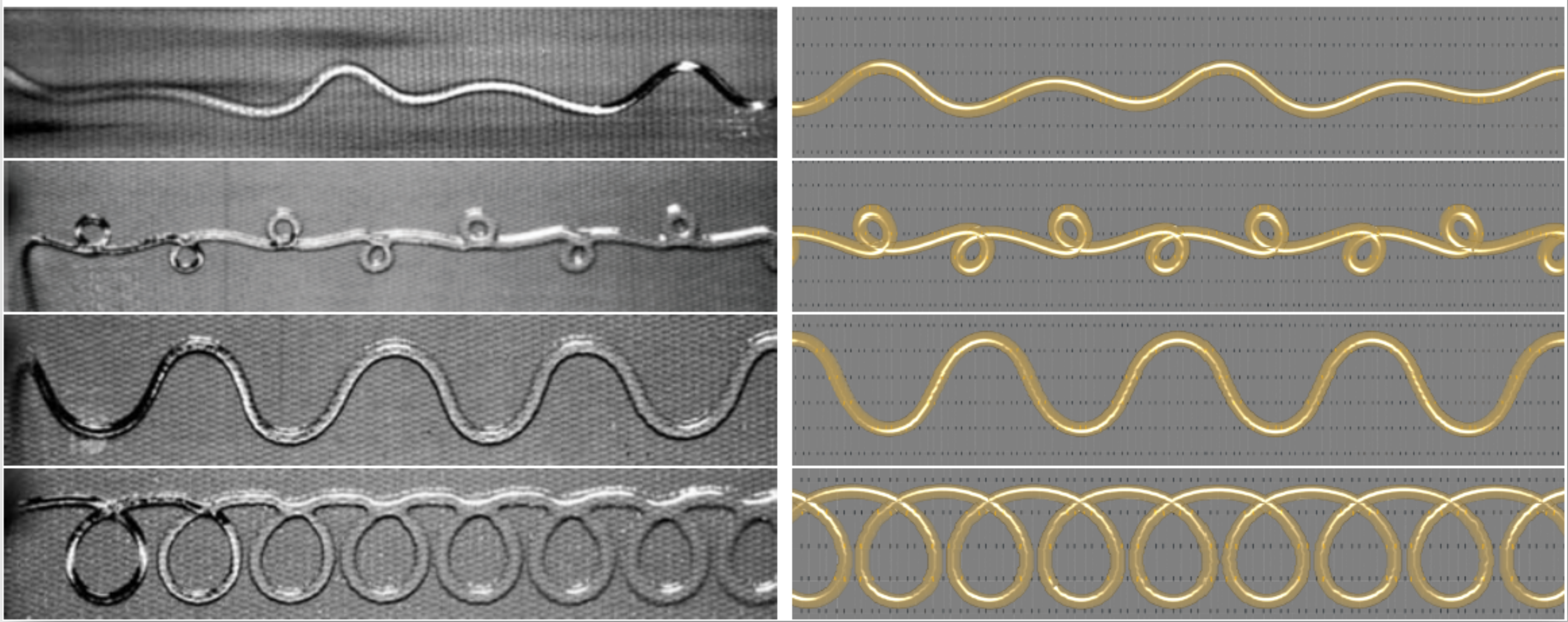
Miklós Bergou
Columbia University

Basile Audoly
UPMC Univ. Paris 06 & CNRS

Etienne Vouga
Columbia University

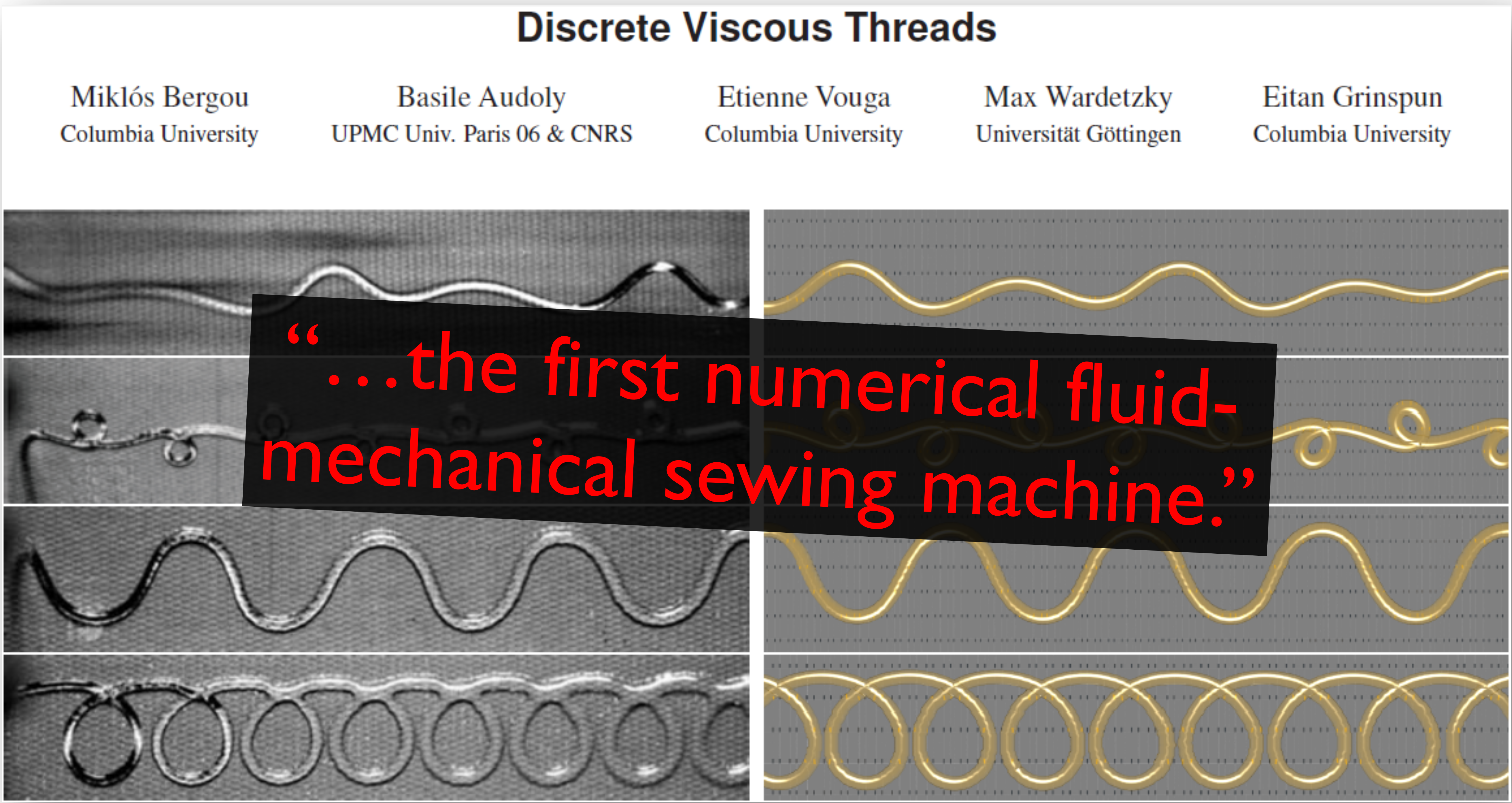
Max Wardetzky
Universität Göttingen

Eitan Grinspun
Columbia University



http://www.cs.columbia.edu/cg/threads/

# Extension and Speedup




http://www.cs.columbia.edu/cg/threads/

# Morals

One curve,
three curvatures.

$$\theta \qquad\qquad 2\sin\frac{\theta}{2} \qquad\qquad 2\tan\frac{\theta}{2}$$

# Morals

**Easy theoretical object, hard to use.**

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

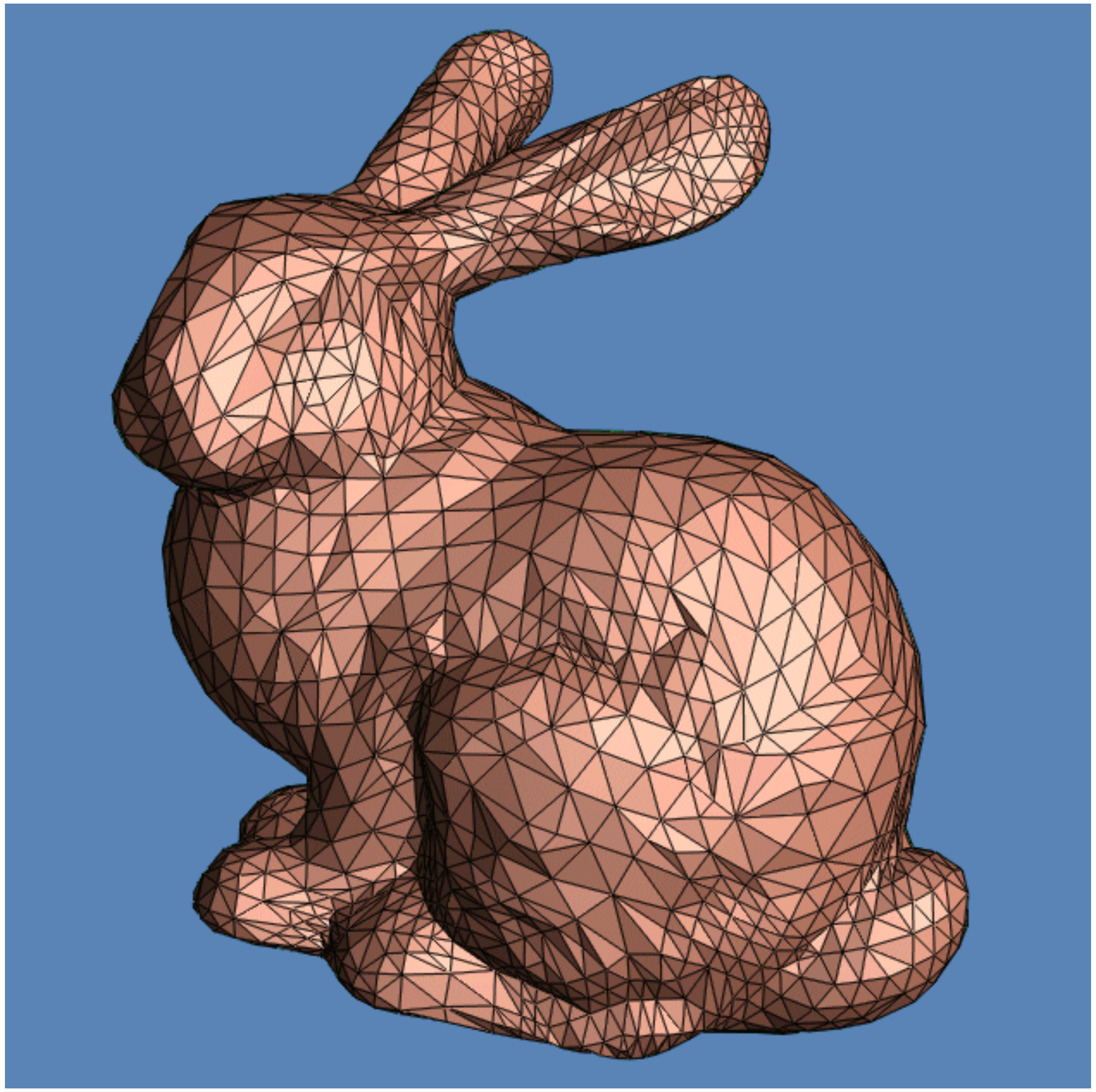
# Morals

Proper frames and DOFs go a long way.

$$m_1^i = u^i \cos\theta^i + v^i \sin\theta^i$$

$$m_2^i = -u^i \sin\theta^i + v^i \cos\theta^i$$

# Next



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

# Surfaces