UC San Diego

Curves: Parametrization, Curvature, Frenet Frame

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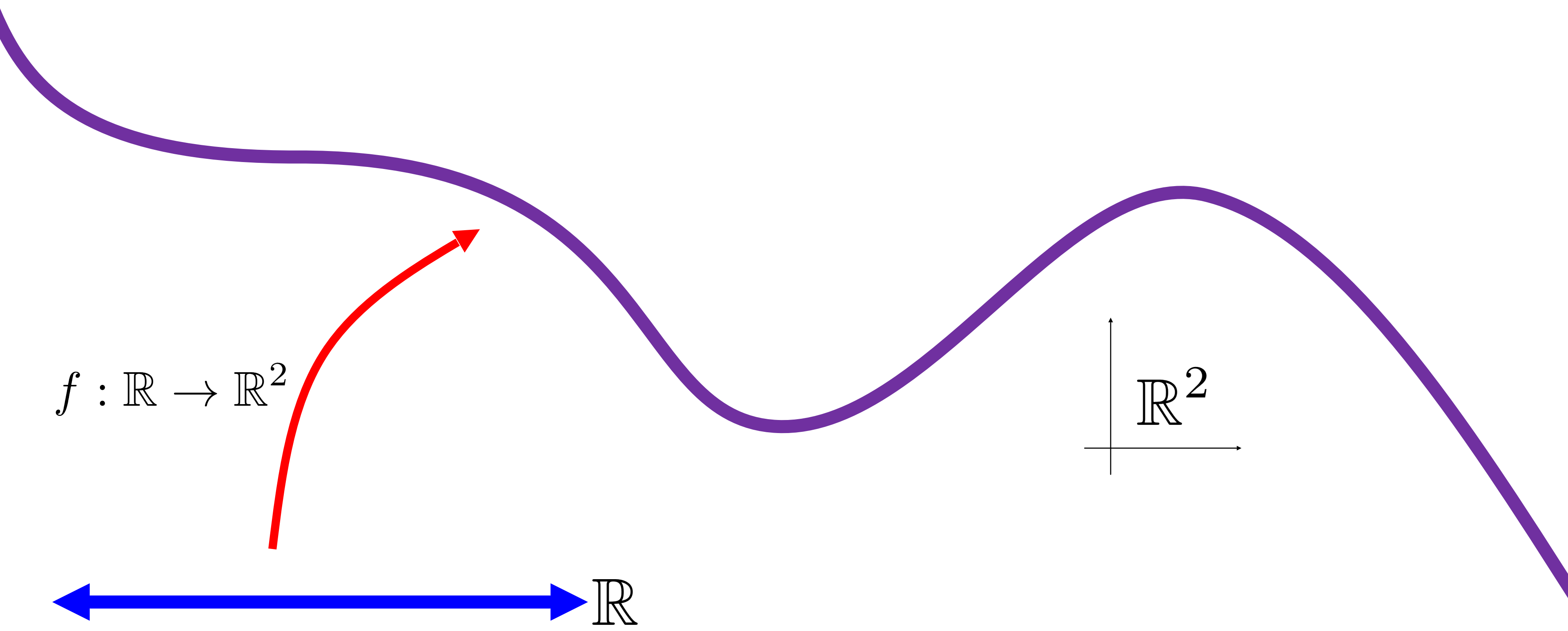
Credit: Justin Solomon

Defining “Curve”



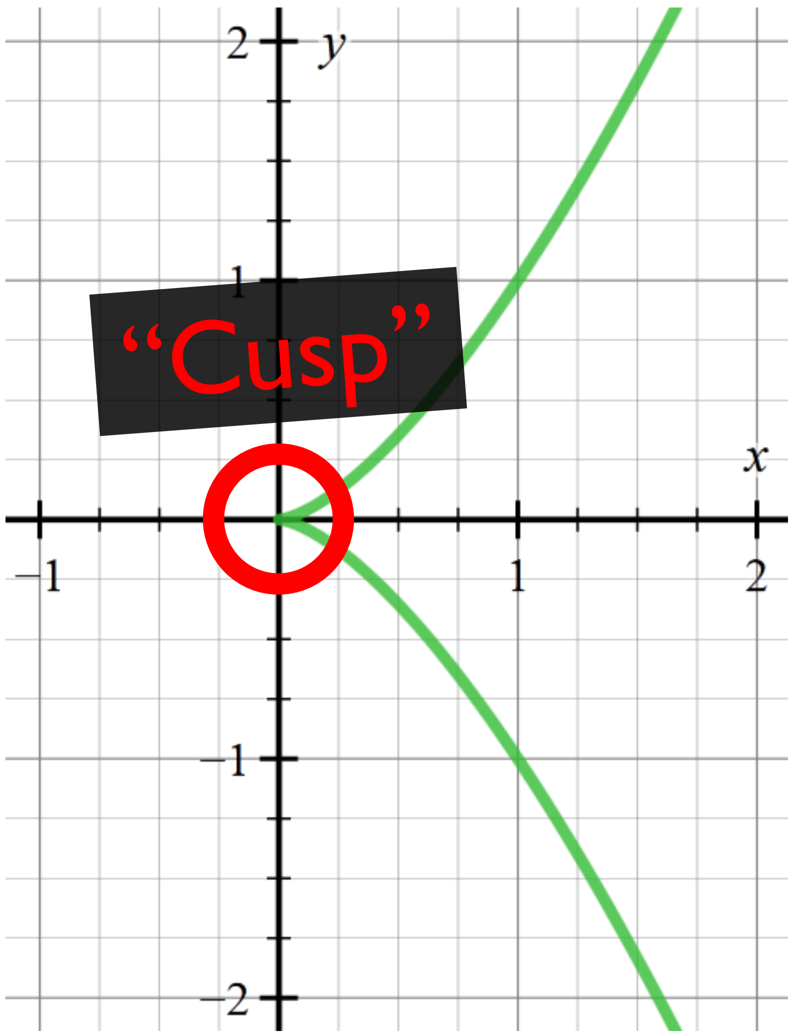
A function?

Subtlety

$$\gamma_3(t) := (0, 0)$$

Not a curve

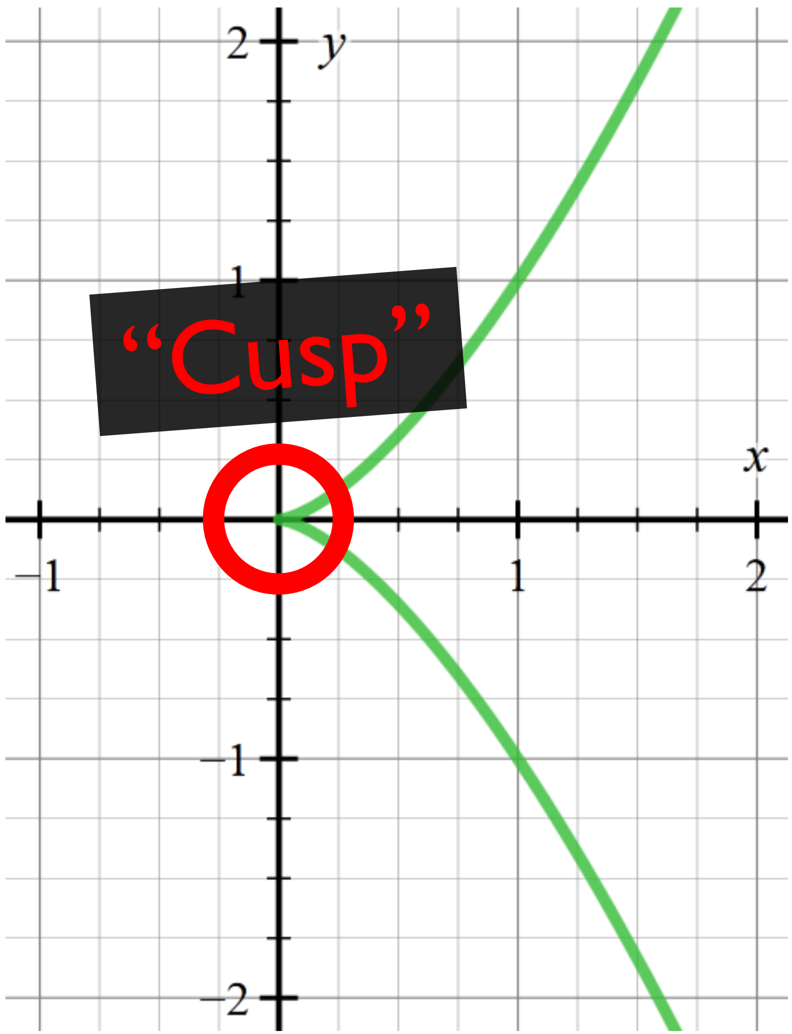
Graphs of Smooth Functions



$$f(t) = (t^2, t^3)$$

http://en.wikipedia.org/wiki/Singular_point_of_a_curve

Graphs of Smooth Functions



$$f(t) = (t^2, t^3)$$

How to ensure the smoothness of a curve?

http://en.wikipedia.org/wiki/Singular_point_of_a_curve

Geometry of a Curve

A curve is a

set of points

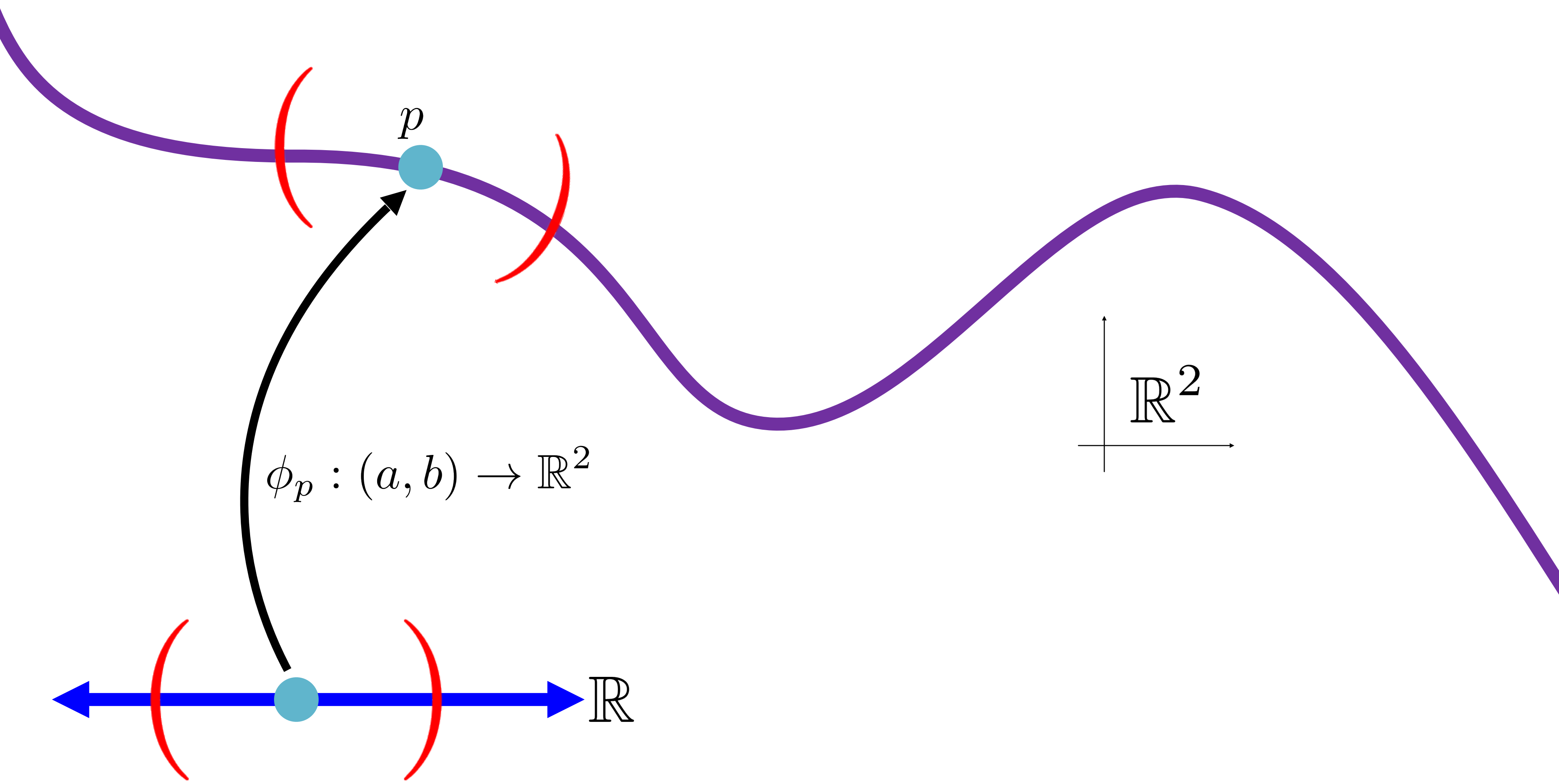
with certain properties.

It is not a function.

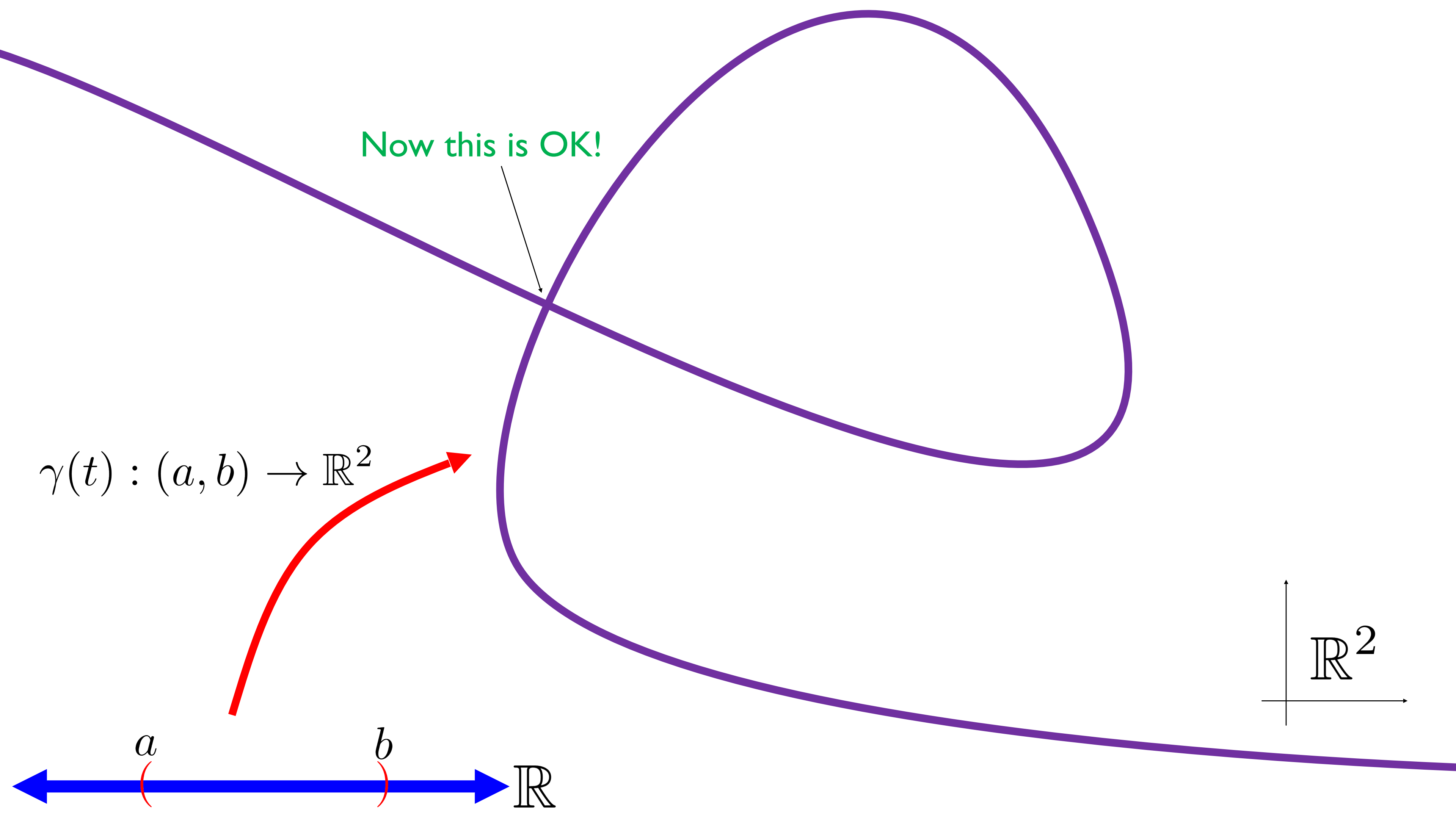
Geometric Definition

Set of points that locally looks like a line.

Differential Geometry Definition



Parameterized Curve



Some Vocabulary

- Trace of parameterized curve

$$\{\gamma(t) : t \in (a,b)\}$$

- Component functions

$$\gamma(t) = (x(t), y(t), z(t))$$

Change of Parameter

$$\bar{t} \mapsto \gamma(g(\bar{t})) = \gamma \circ g(\bar{t})$$

Geometric measurements should be

invariant

to changes of parameter.

Dependence of Velocity

$$\tilde{\gamma}(s) := \gamma(\phi(s))$$

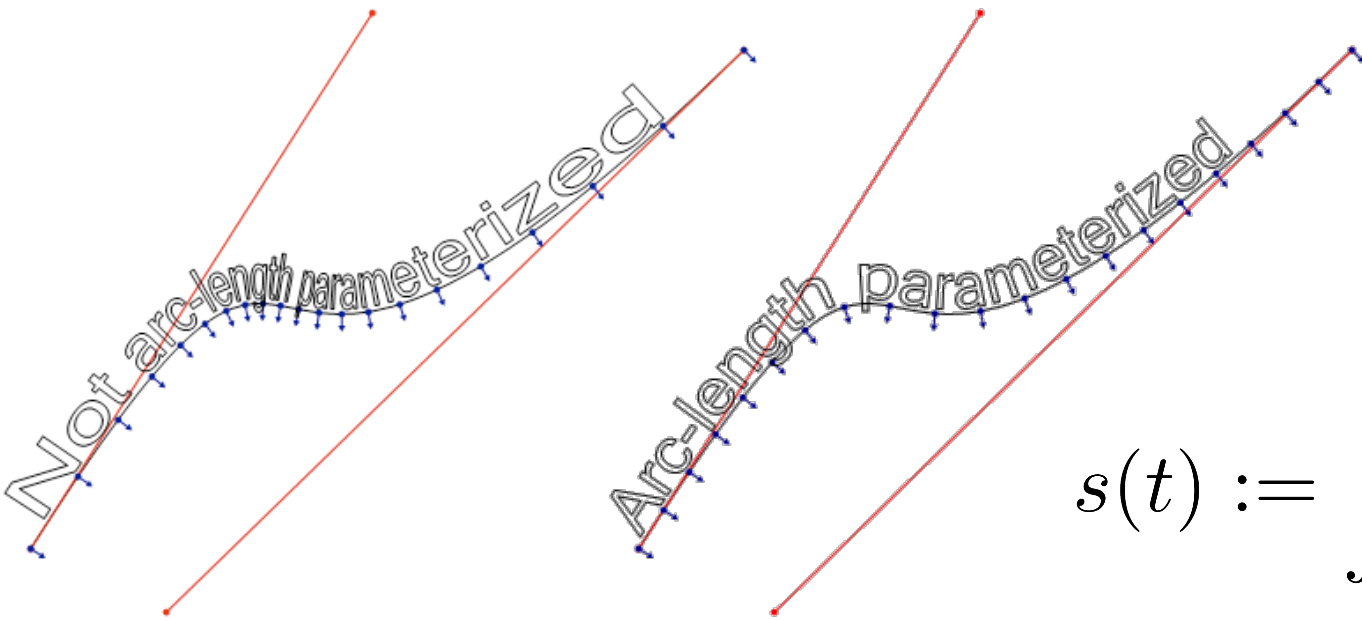
On the board:
Effect on velocity and acceleration.

Arc Length

$$\int_a^b \|\gamma'(t)\|\,dt$$

Parameterization by Arc Length

http://www.planetclegg.com/projects/WarpingTextToSplines.html



$$s(t) := \int_{t_0}^{t} \|\gamma'(t)\| \, dt$$

$$t(s) := \text{inverse of } s(t)$$

$$\bar{\gamma}(s) = \gamma(t(s))$$

Constant-speed parameterization

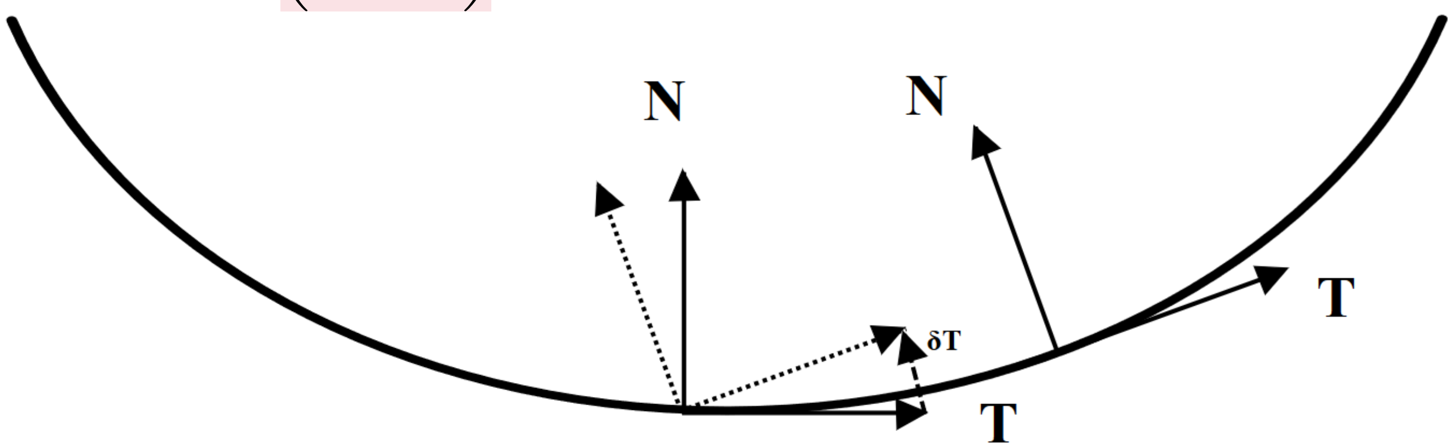
Moving Frame in 2D

$$T(s) := \gamma'(s)$$

$$\implies \text{(on board)} \; \|T(s)\| \equiv 1$$

$$N(s) := JT(s)$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Philosophical Point

Differential geometry "should" be

coordinate-invariant.

Referring to x and y is a hack!

(but sometimes convenient…)

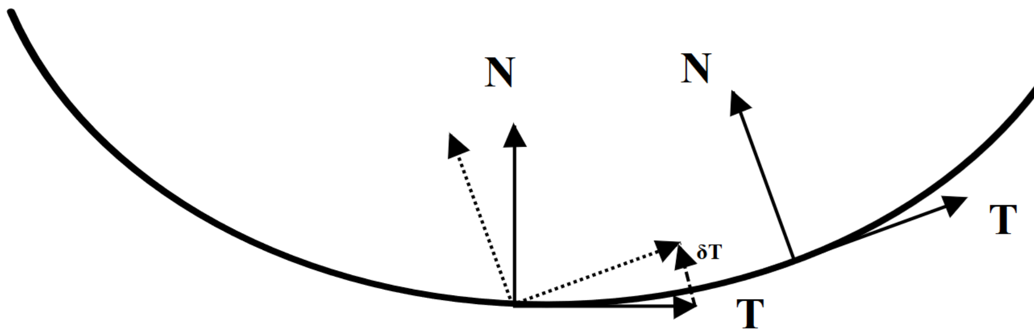
Q:

How do you characterize shape without coordinates?

Turtles All The Way Down

On the board:

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$



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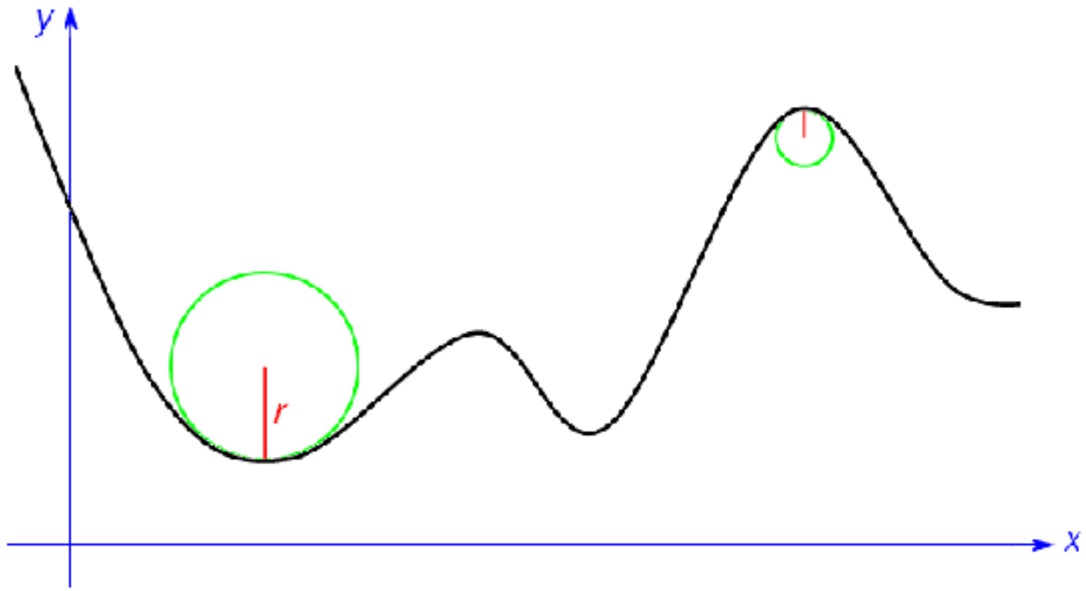
Use coordinates *from* the curve to express its shape!

Radius of Curvature



$$r(s) := \frac{1}{k(s)}$$

https://www.quora.com/What-is-the-base-difference-between-radius-of-curvature-and-radius-of-gyration

Invariance is Important

Fundamental theorem of the local theory of plane curves:

$k(s)$ characterizes a planar curve up to rigid motion.

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Fundamental theorem of the local theory of plane curves:

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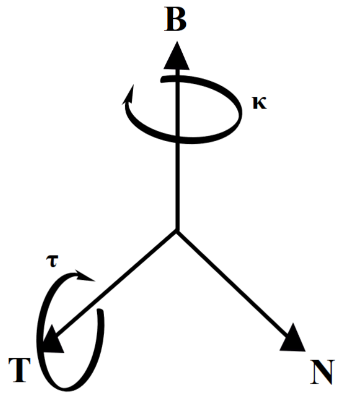
↑

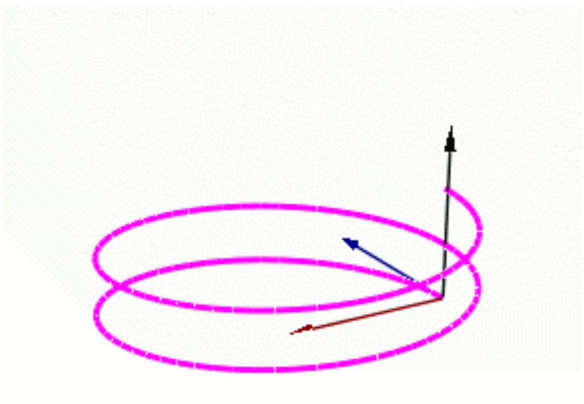
Statement shorter than the name!

Frenet Frame: Curves in $\mathbb{R}^3$

- Binormal:
- Curvature: In-plane motion
- Torsion: Out-of-plane motion

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$





Fundamental theorem of the local theory of space curves:

Curvature and torsion characterize a 3D curve up to rigid motion.

Aside: Generalized Frenet Frame

$$\gamma(s) : \mathbb{R} \to \mathbb{R}^n$$

$$\frac{d}{ds}\begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix} = \begin{pmatrix} 0 & \chi_1(s) & & 0 \\ -\chi_1(s) & \ddots & \ddots & \\ & \ddots & 0 & \chi_{n-1}(s) \\ 0 & & -\chi_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_n(s) \end{pmatrix}$$

Suspicion: Application to time series analysis? ML?

C. Jordan, 1874

Gram-Schmidt on first *n* derivatives