UC San Diego

Laplacian

(Graph Embedding, Heat Kernel Signature, Continuous Theory)

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LAPLACIAN GRAPH EMBEDDING

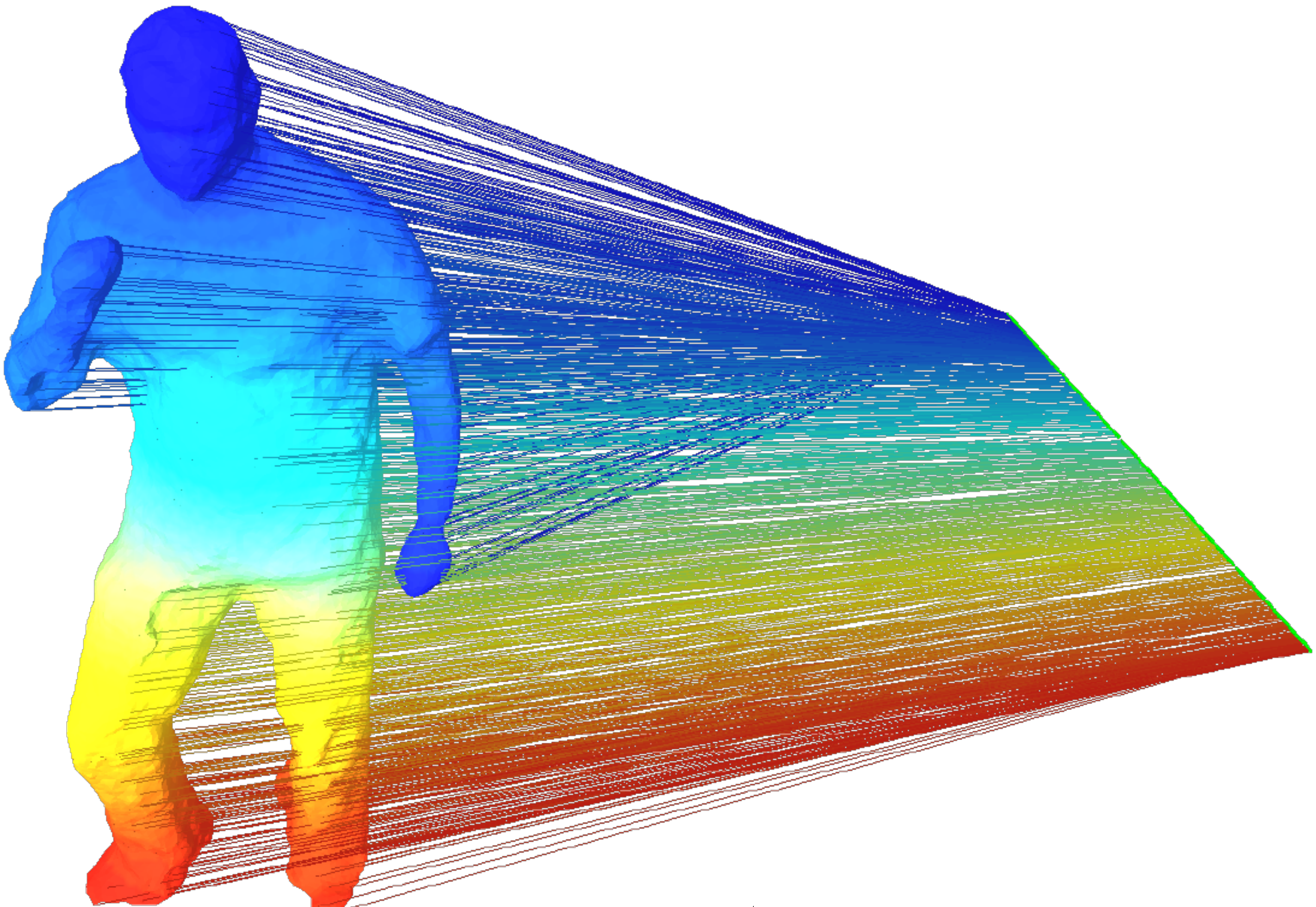
1-d Laplacian Embedding

- Map a weighted graph onto a line such that connected nodes stay as close as possible, i.e., minimize $\sum_{i,j=1}^{n} w_{ij}(f(v_i) - f(v_j))^2$, or:

$$\arg\min_{\boldsymbol{f}} \boldsymbol{f}^\top \mathbf{L} \boldsymbol{f} \text{ with: } \boldsymbol{f}^\top \boldsymbol{f} = 1 \text{ and } \boldsymbol{f}^\top \mathbf{1} = 0$$

- The solution is the eigenvector associated with the smallest nonzero eigenvalue of the eigenvalue problem: $\mathbf{L}\boldsymbol{f} = \lambda \boldsymbol{f}$, namely the Fiedler vector $\boldsymbol{u}_2$.
- For more details on this minimization see Golub & Van Loan *Matrix Computations*, chapter 8 (The symmetric eigenvalue problem).

1-d Embedding Example



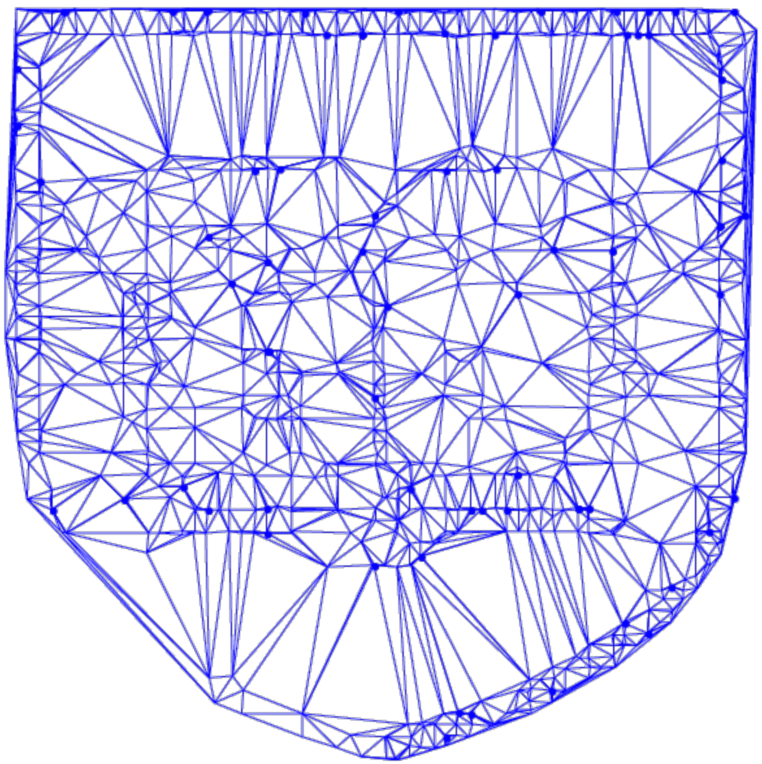
Higher-d Embeddings

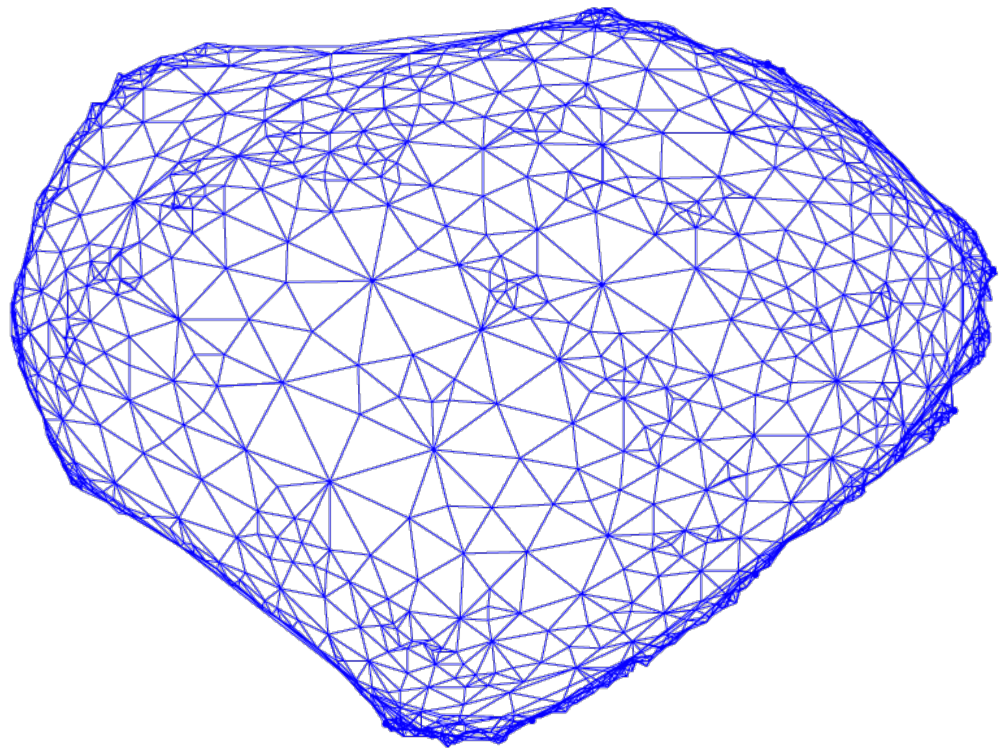
- Embed the graph in a k-dimensional Euclidean space. The embedding is given by the $n \times k$ matrix $\mathbf{F} = [\boldsymbol{f}_1 \boldsymbol{f}_2 \dots \boldsymbol{f}_k]$ where the i-th row of this matrix – $\boldsymbol{f}^{(i)}$ – corresponds to the Euclidean coordinates of the i-th graph node v_i.
- We need to minimize (Belkin & Niyogi '03):

$$\arg \min_{\boldsymbol{f}_1 \dots \boldsymbol{f}_k} \sum_{i,j=1}^{n} w_{ij} \|\boldsymbol{f}^{(i)} - \boldsymbol{f}^{(j)}\|^2 \text{ with: } \mathbf{F}^\top \mathbf{F} = \mathbf{I}.$$

- The solution is provided by the matrix of eigenvectors corresponding to the k lowest nonzero eigenvalues of the eigenvalue problem $\mathbf{L}\boldsymbol{f} = \lambda \boldsymbol{f}$.

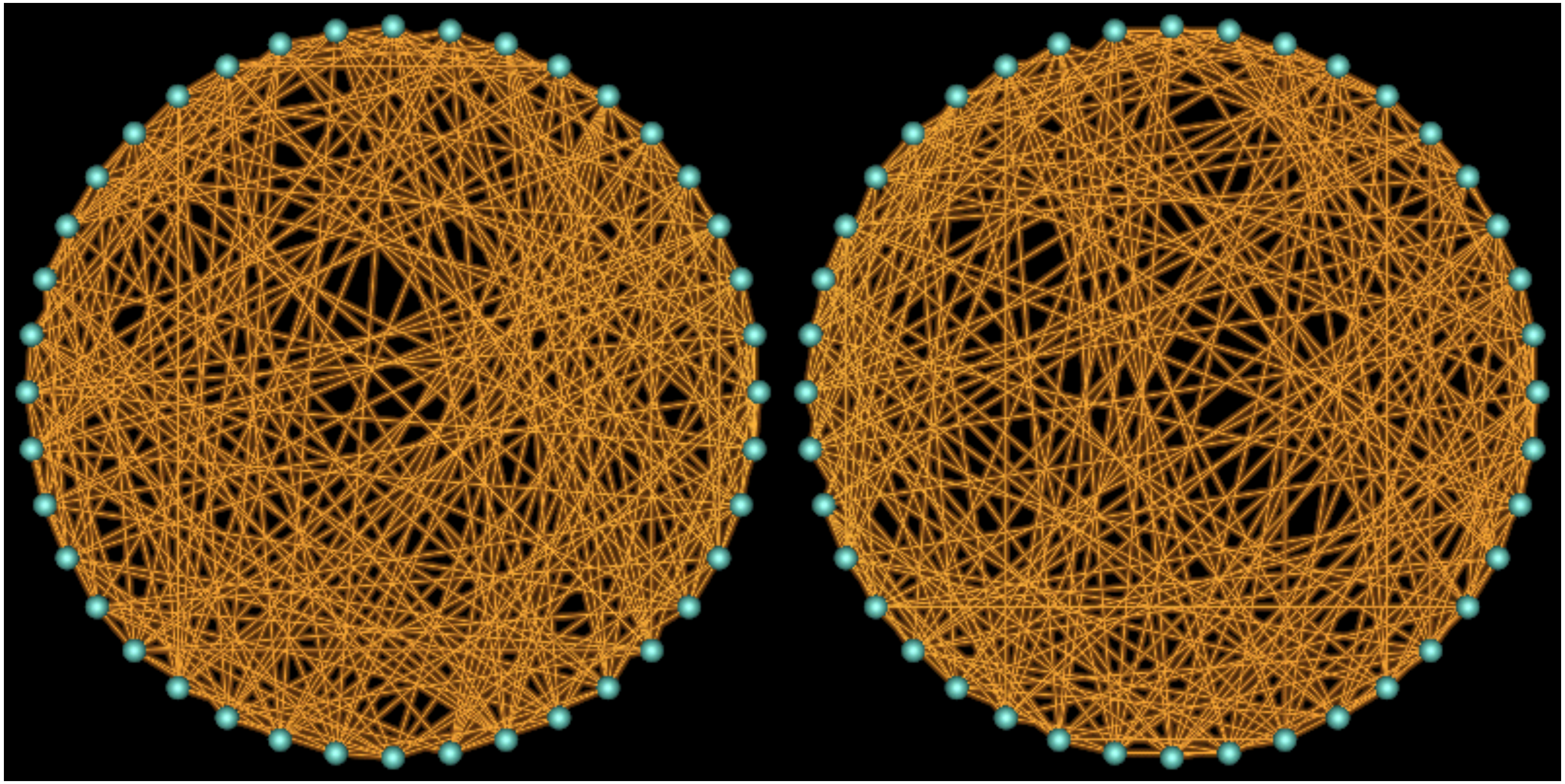
2-d Embeddings



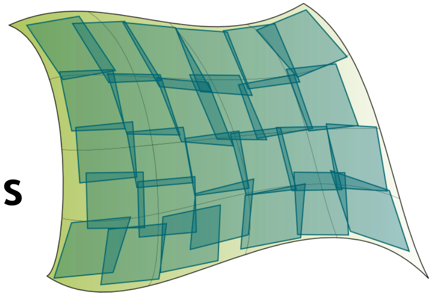


LAPLACIAN FOR SHAPE DESCRIPTOR

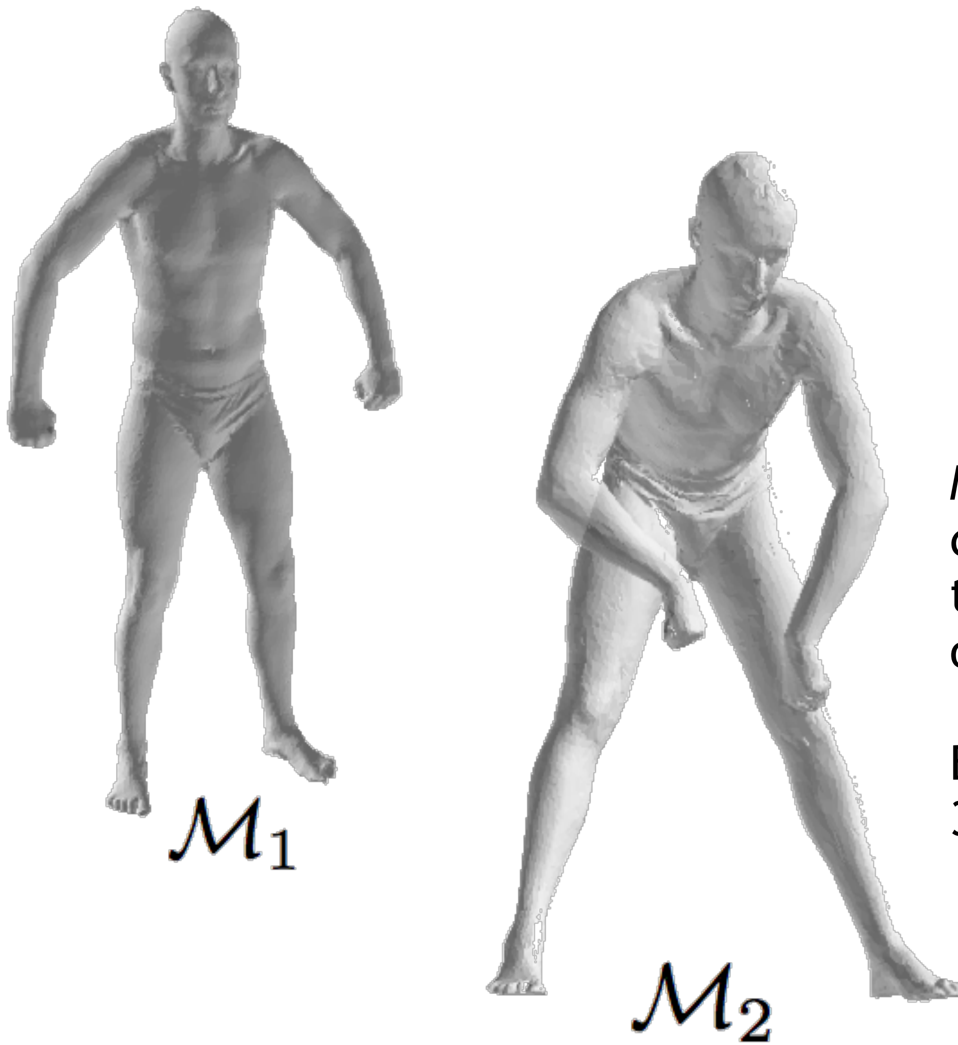
Graph Isomorphism



Intrinsic alignment of **manifolds**



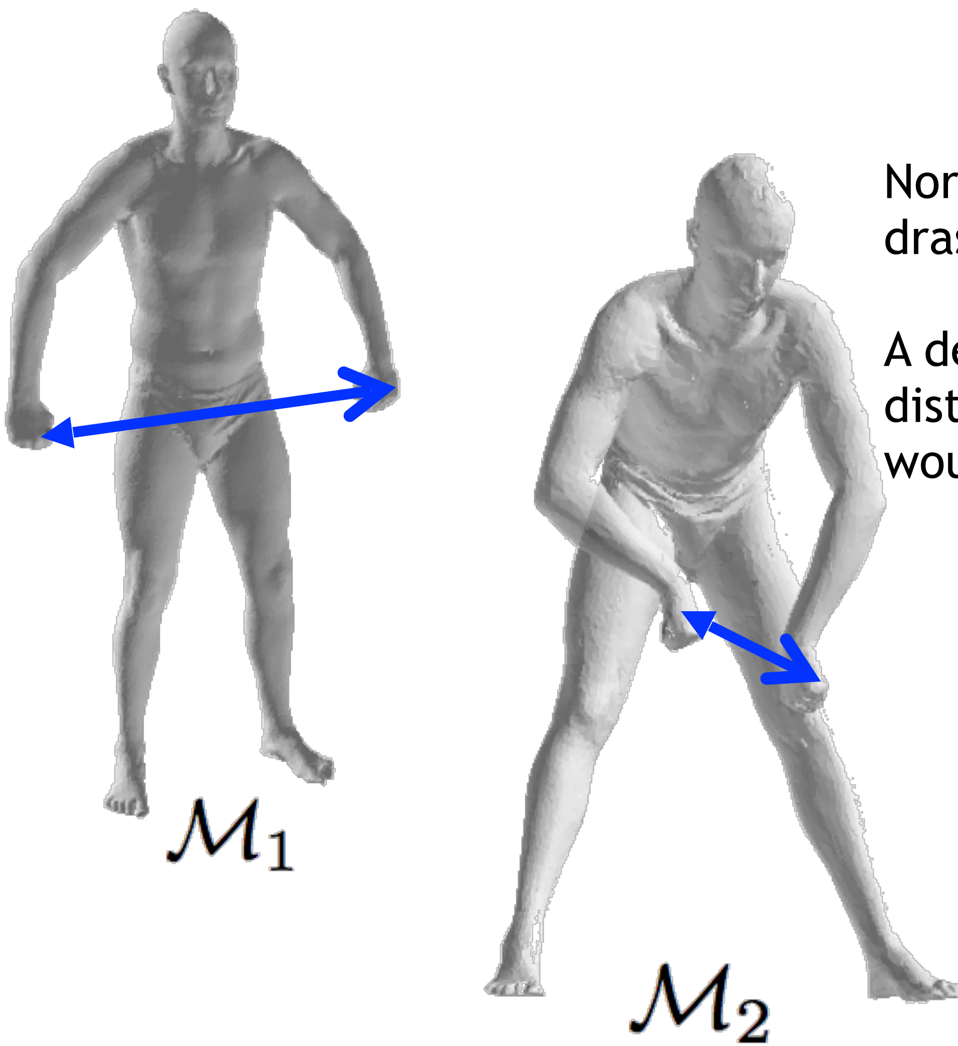
Why Intrinsic?



Many shapes have natural deformations and articulations that do not change the nature of the shape.

But they change its embedding 3D space.

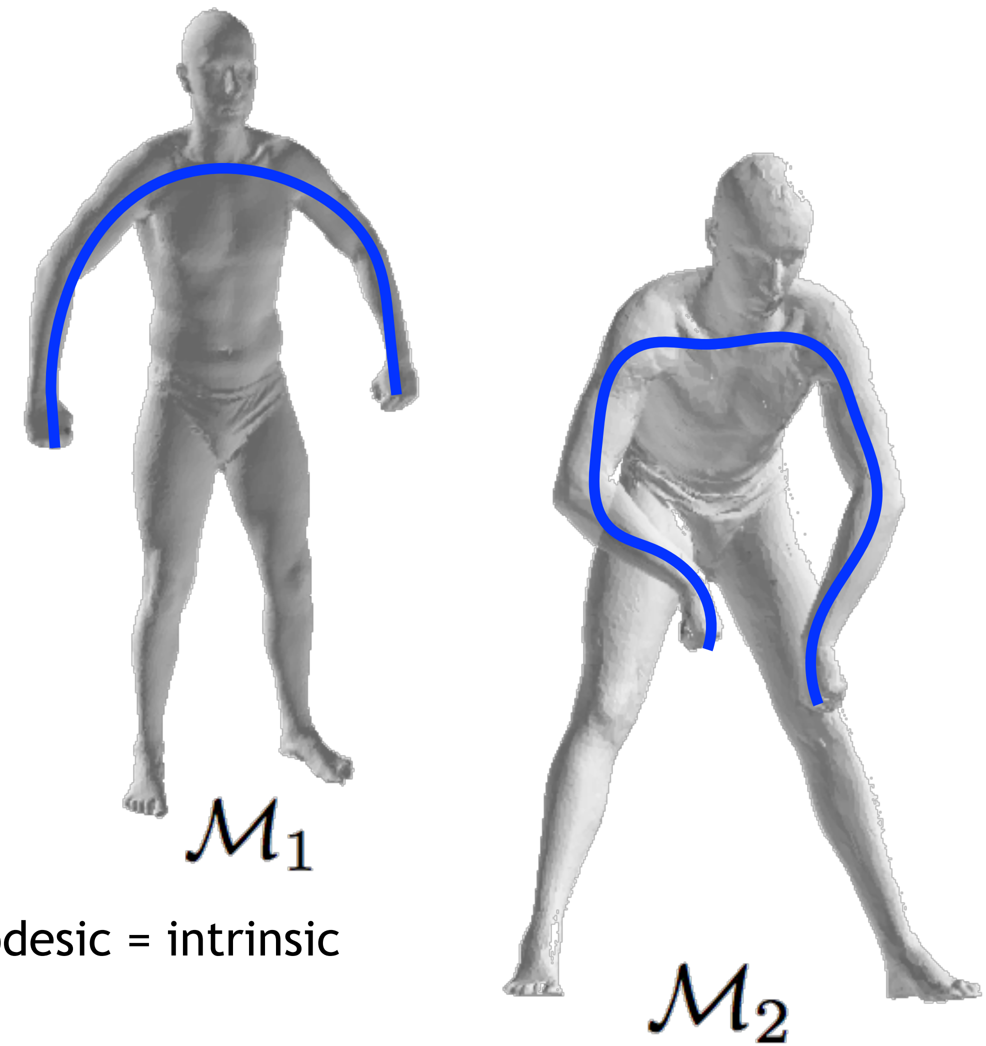
Why Intrinsic?



Normal distances can change drastically under such deformations

A descriptor based on Euclidean distance histograms, like D2, would fail

Geodesic / Intrinsic Distances



Near isometric deformations are common for both organic and man-made shapes

Intrinsic distances are invariant to isometric deformations

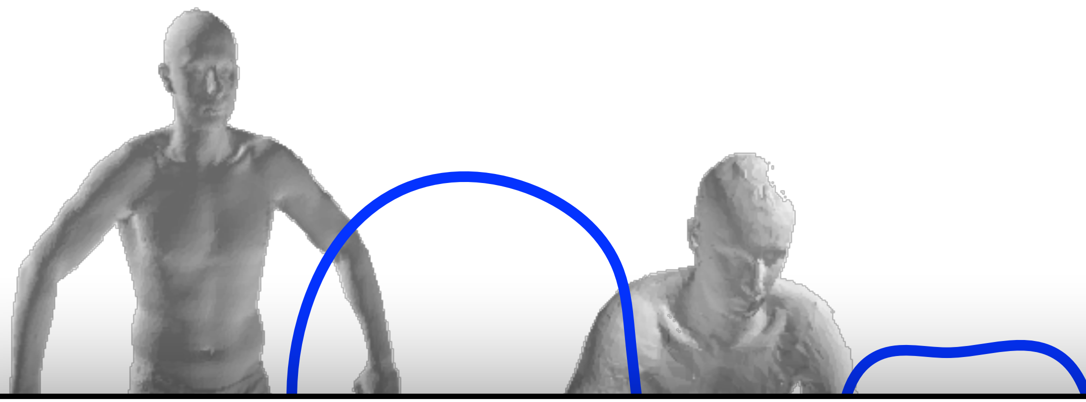
geodesic = intrinsic

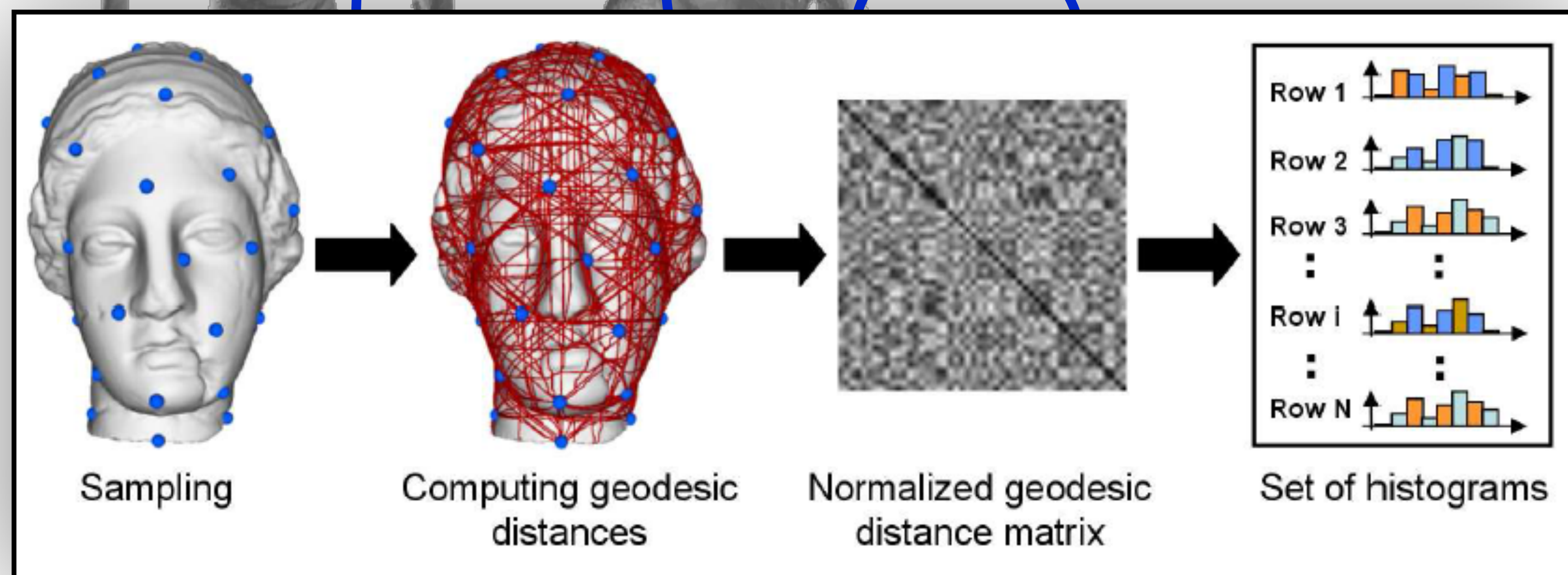
isometry = length-preserving transform

No stretching, shrinking, or tearing

Geodesic / Intrinsic Distances

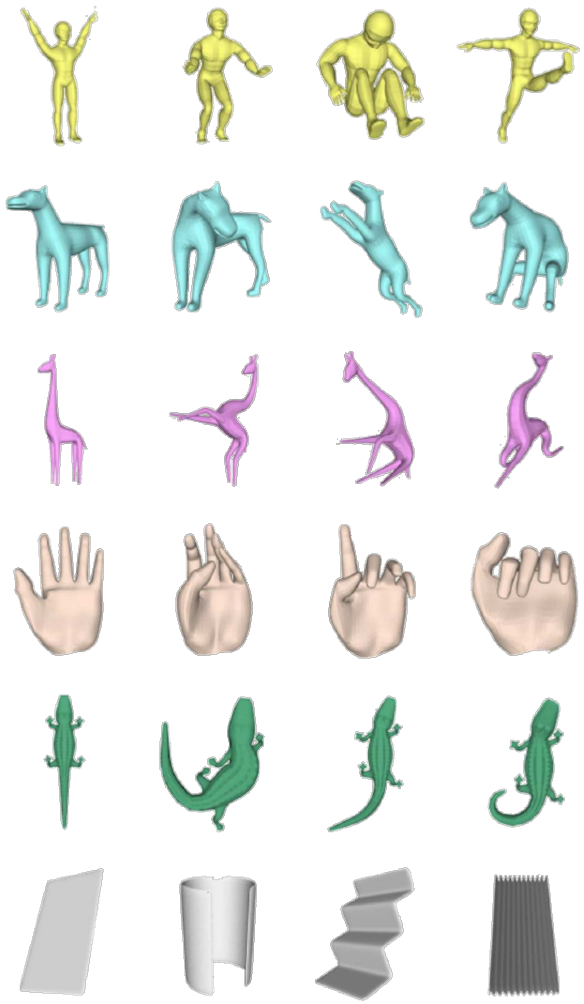
We can use geodesic distance histograms

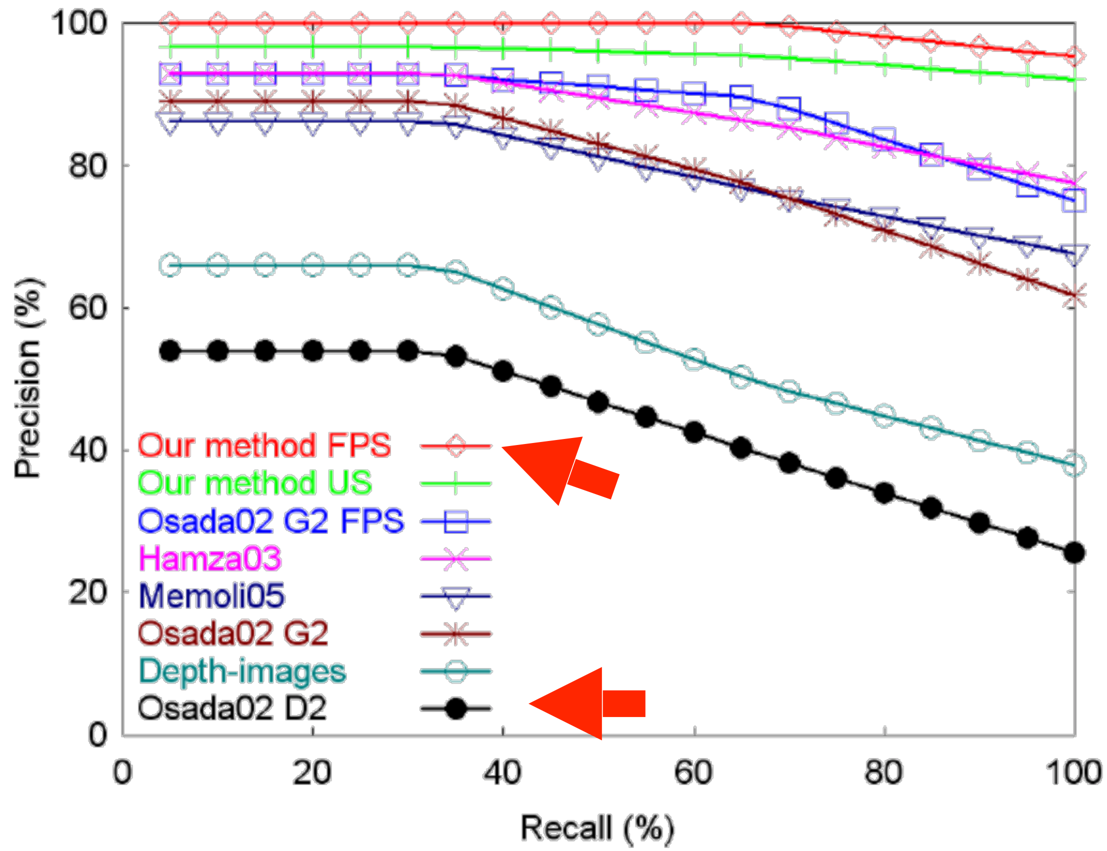




Ruggeri et al. 2008

Geodesic / Intrinsic Distances

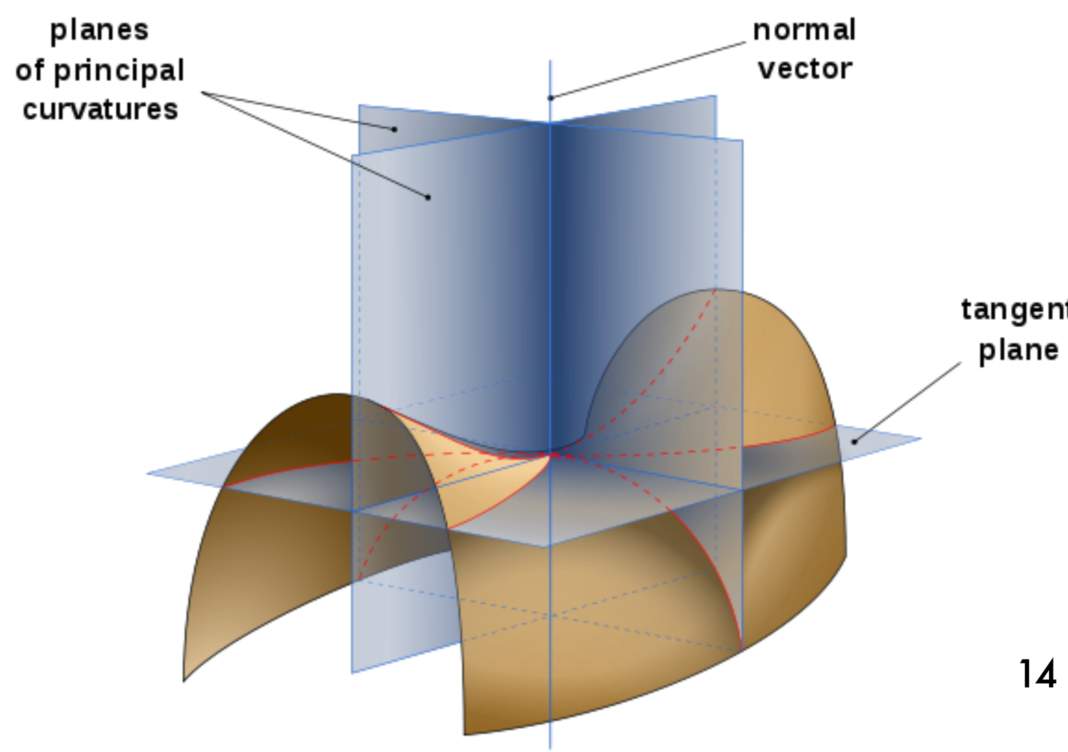




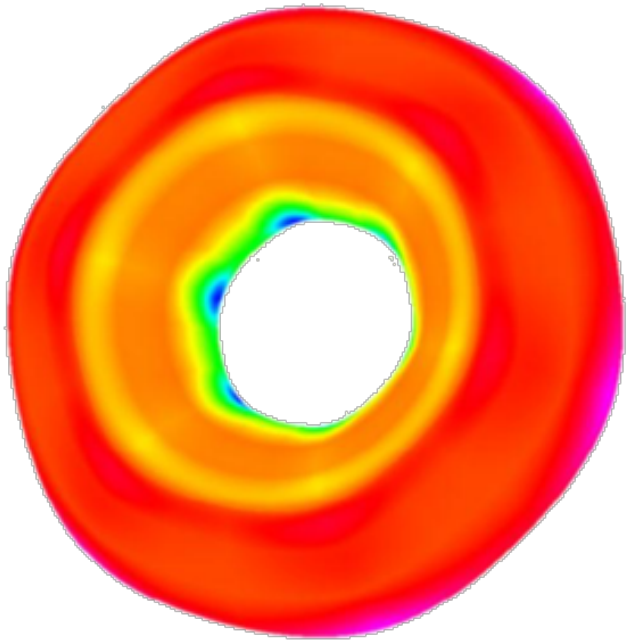
Ruggeri et al. 2008

What About Local Intrinsic Descriptors?

- Isometrically invariant features
  - Curvature
  - Geodesic Distance
  - Histogram of Geodesic Distances (similar to D2)
  - Global Point Signature
  - Heat Kernel Signature
  - Wave Kernel Signature



Gaussian Curvature



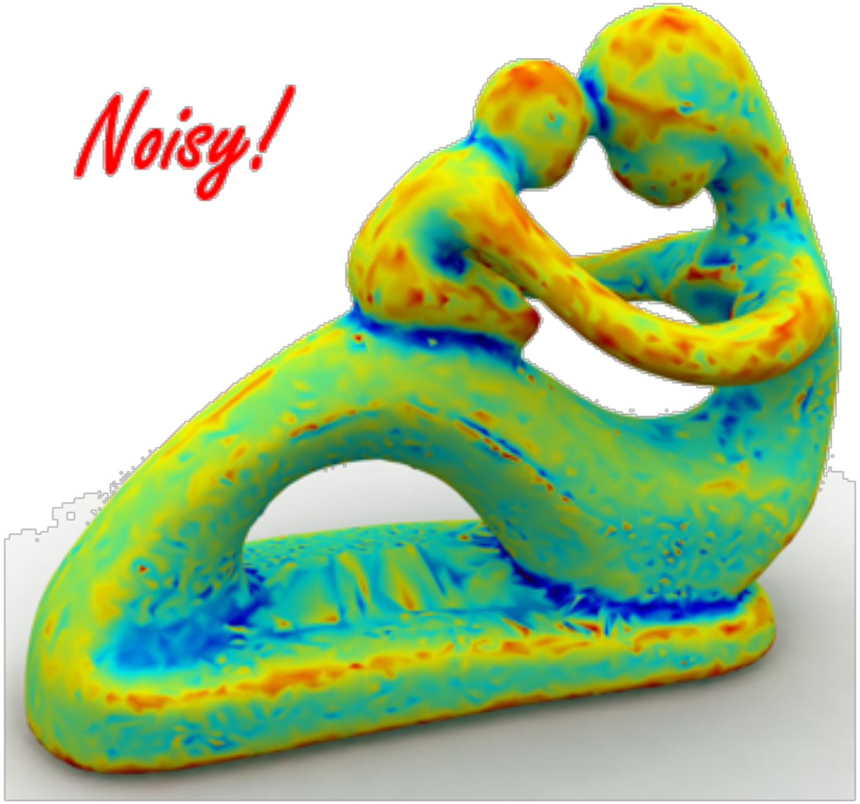
$$K = \kappa_1 \kappa_2$$

Theorema Egregium
("Remarkable Theorem"):
Gaussian curvature is intrinsic.

http://www.sciencedirect.com/science/article/pii/S0010448510001983

Gaussian Curvature

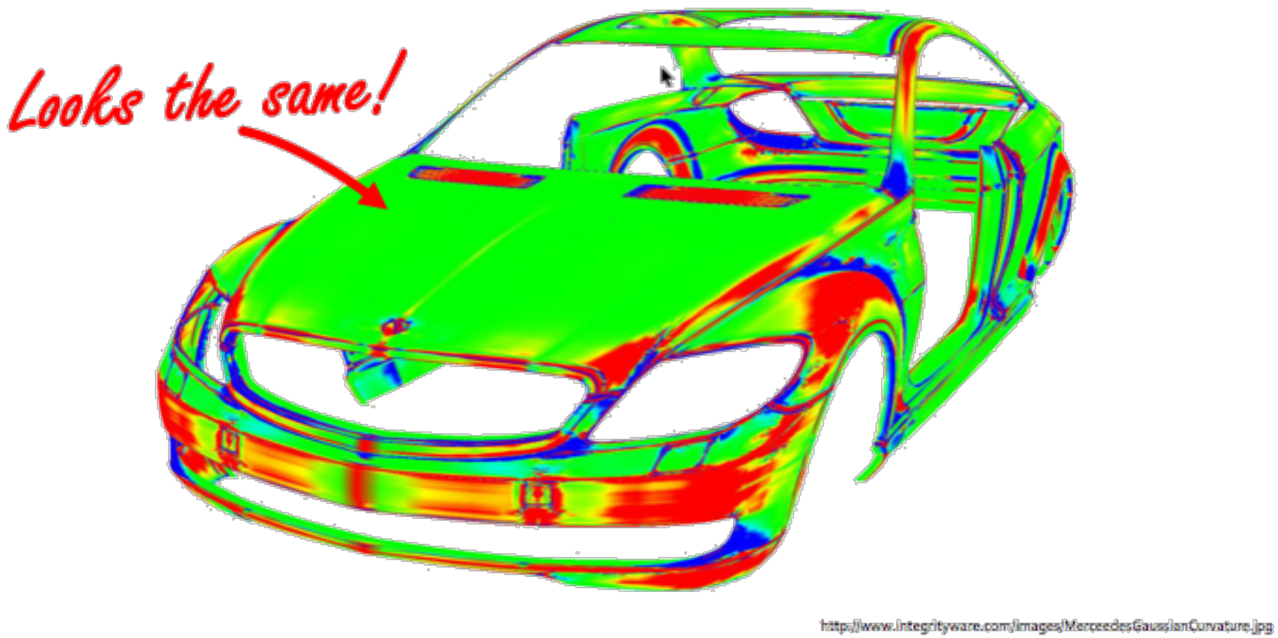
Problems



$$K = \kappa_1 \kappa_2$$

Gaussian Curvature

Problems



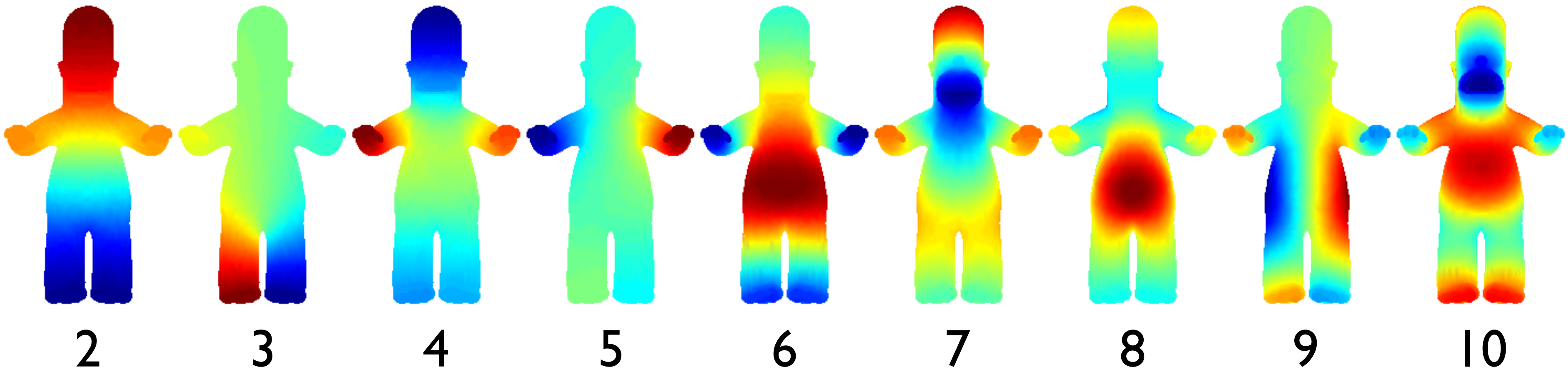
$$K = \kappa_1 \kappa_2$$

Solomon

Intrinsic Observation

Heat diffusion patterns are not affected if you bend a surface.

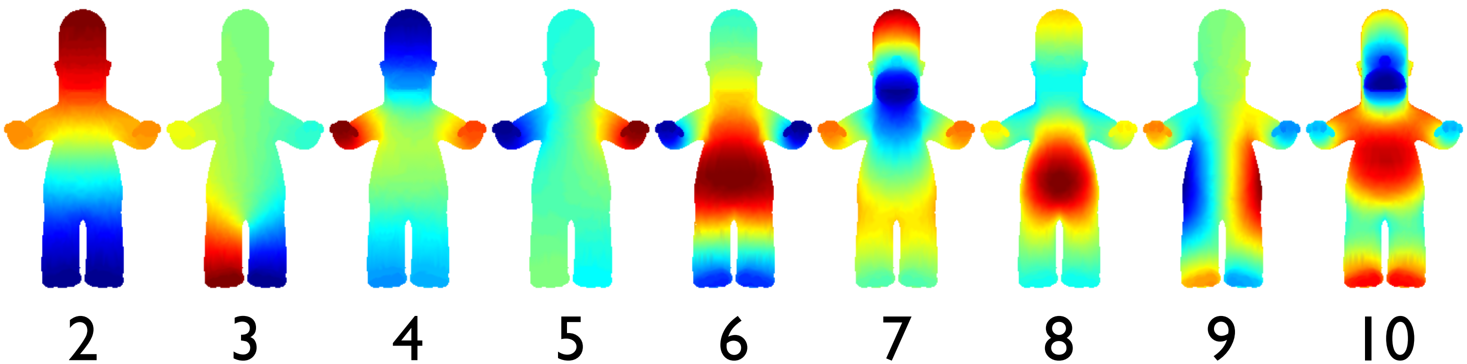
Global Point Signature



$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

"Laplace-Beltrami Eigenfunctions for Deformation Invariant Shape Representation"
Rustamov, SGP 2007

Global Point Signature

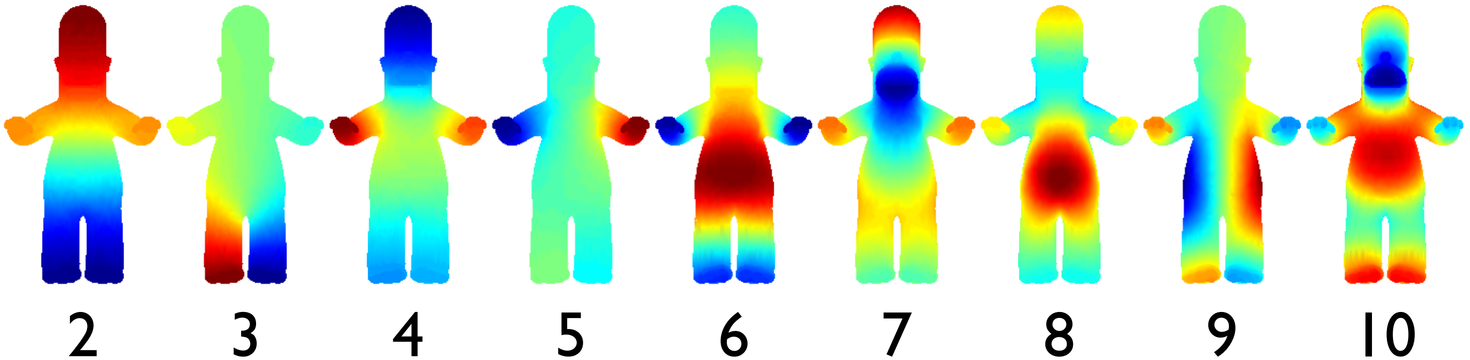


$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

If surface does not self-intersect, neither does the GPS embedding.

Proof: Laplacian eigenfunctions span ; if GPS(p)=GPS(q), then all functions on S would be equal at p and q.

Global Point Signature



$$\mathrm{GPS}(p) := \left(-\frac{1}{\sqrt{\lambda_1}}\phi_1(p), -\frac{1}{\sqrt{\lambda_2}}\phi_2(p), -\frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

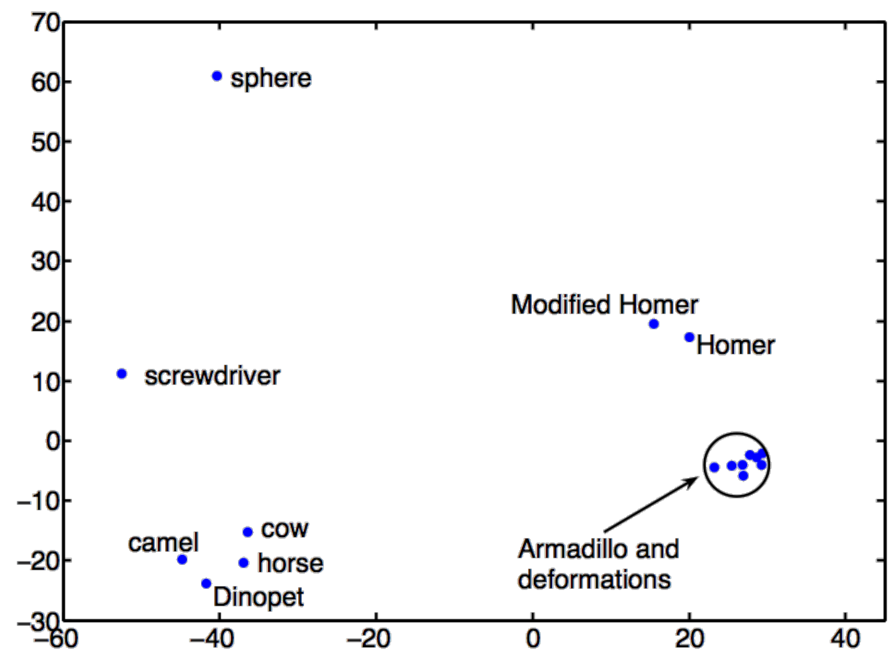
GPS is isometry-invariant.

Proof: Comes from the Laplacian.

Global Point Signature

$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

Figure 4: *Armadillo and its deformations.*



Similar to D2, but use histograms in embedded space (rather than Euclidean)

Rustamov et al. 2007

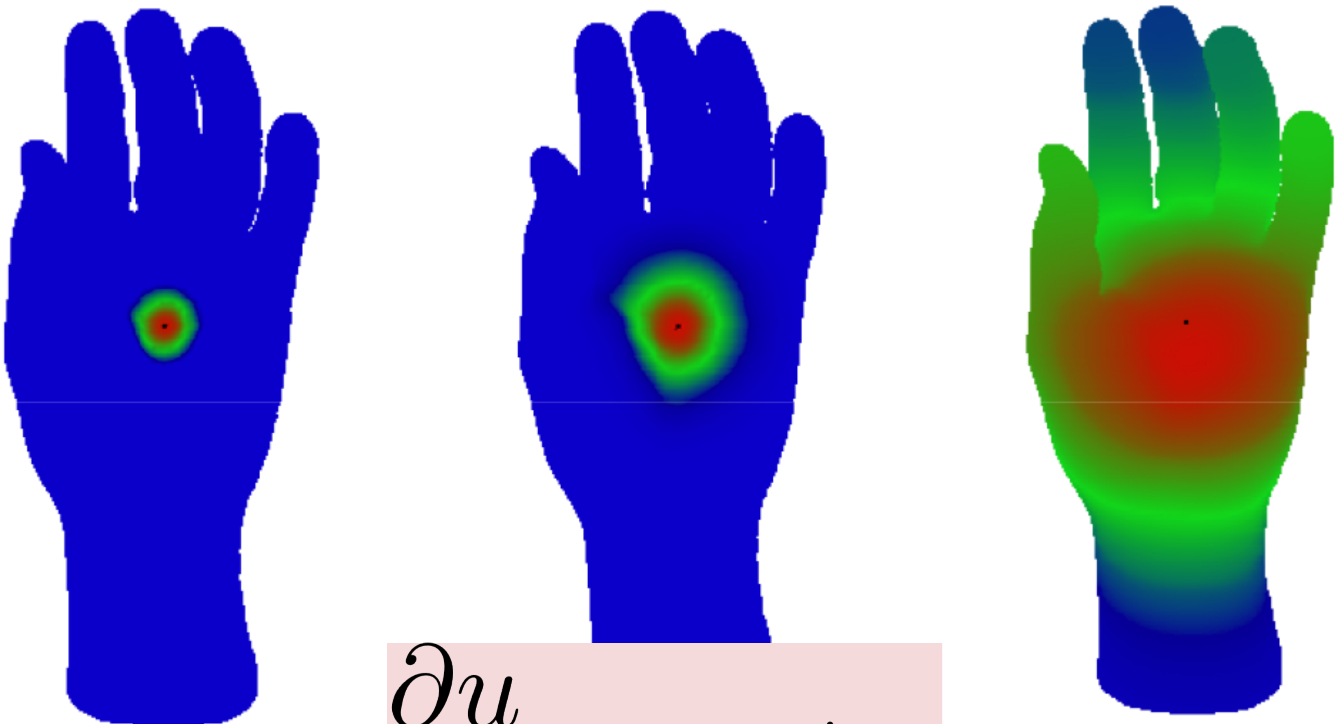
Global Point Signature

$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots \right)$$

- Pros
  - Isometry-invariant
  - Global (each point feature depends on entire shape)
- Cons
  - Eigenfunctions may flip sign
  - Eigenfunctions might change positions due to deformations
  - Only global

Rustamov et al. 2007

Recall: Connection to Physics

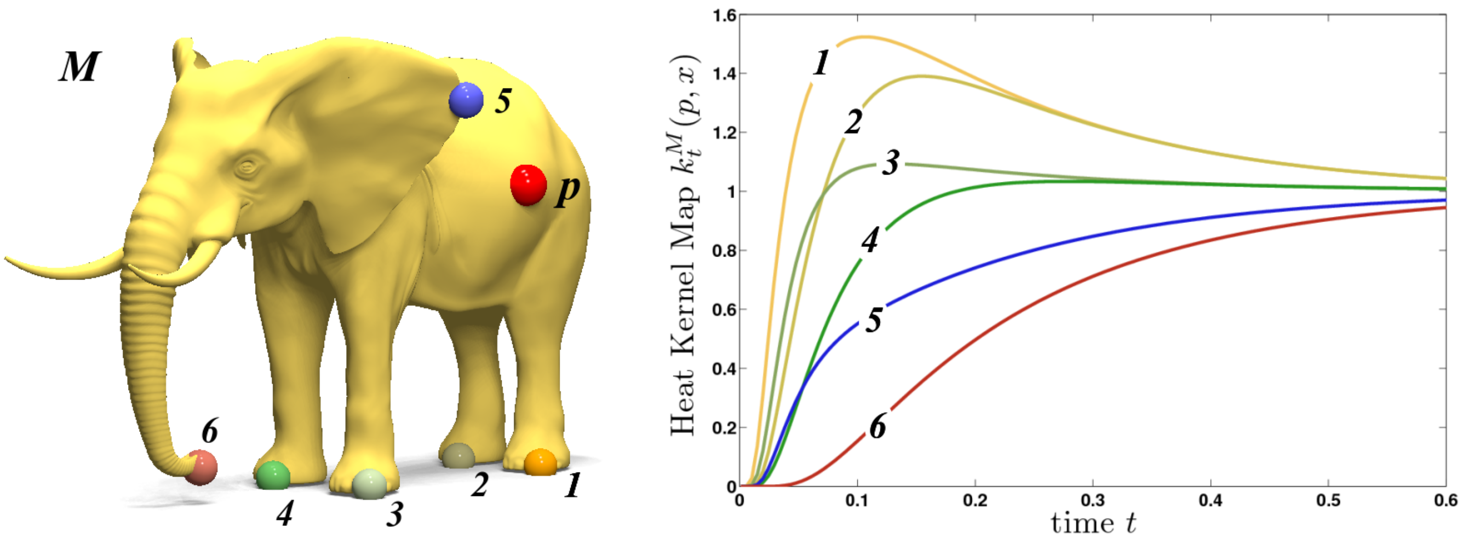


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Heat Kernel Map

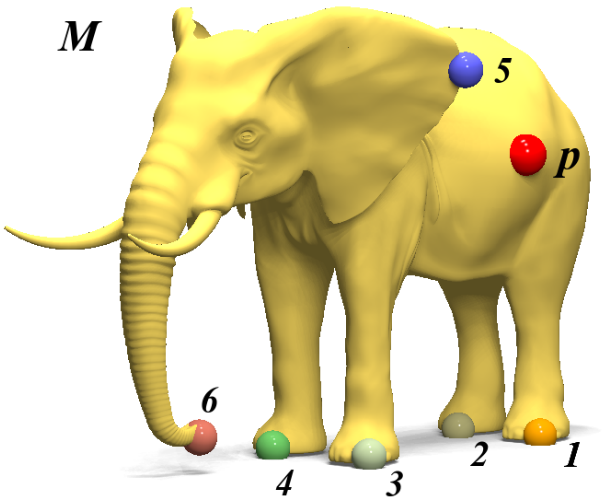


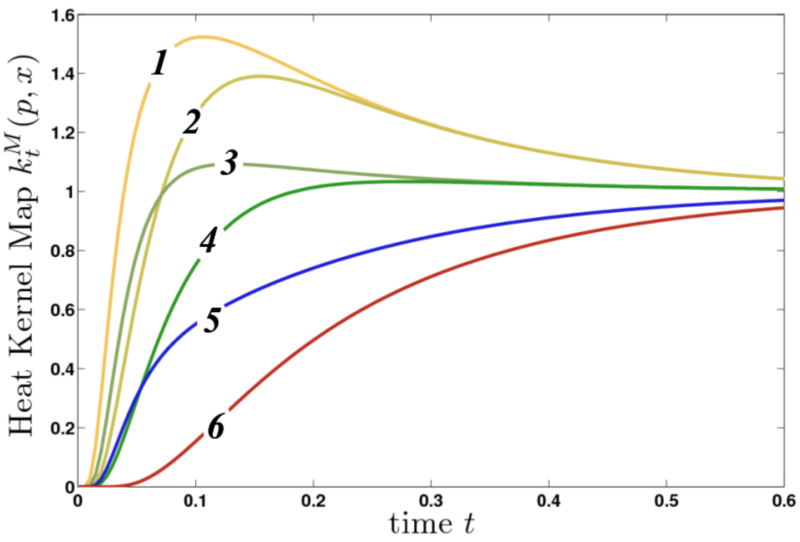
$$\mathrm{HKM}_p(x,t) := k_t(p,x)$$

How much heat diffuses from *p* to *x* in time *t*?

One Point Isometric Matching with the Heat Kernel
Ovsjanikov et al. 2010

Heat Kernel Map





$$\mathrm{HKM}_p(x,t) := k_t(p,x)$$

Theorem: Only have to match one point!

One Point Isometric Matching with the Heat Kernel
Ovsjanikov et al. 2010

KNN