UC San Diego

Laplacian

(Mesh editing, Spectral Graph Theory)

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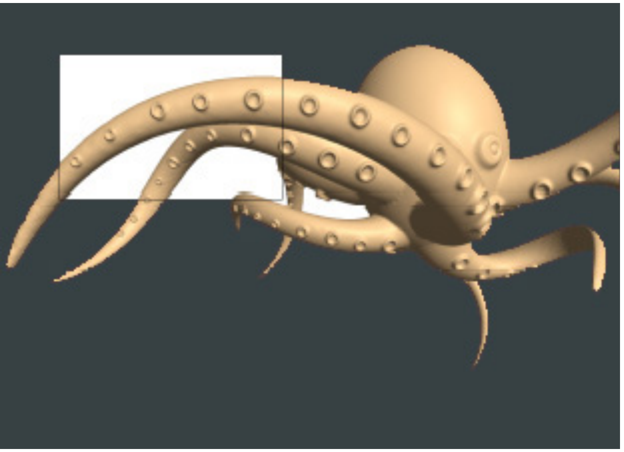
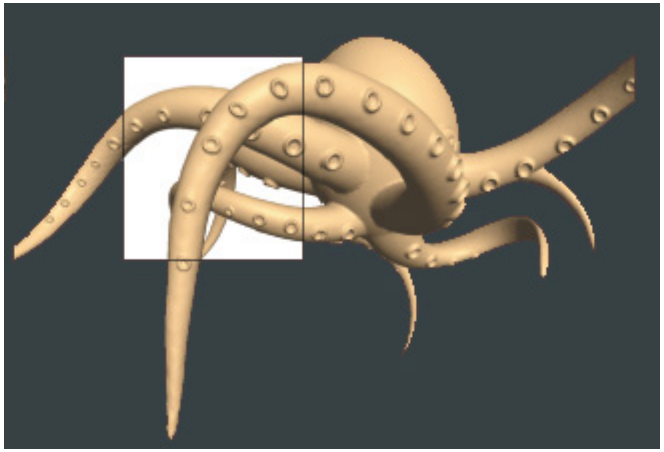
LAPLACIAN MESH EDITING

Our Goal

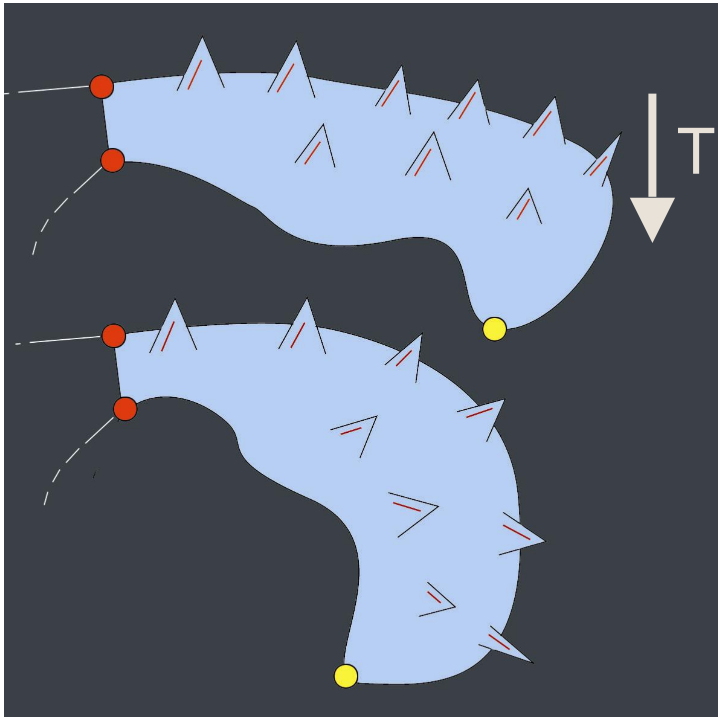
Edit a surface while retaining its visual appearance





Editing a surface while retaining its visual appearance

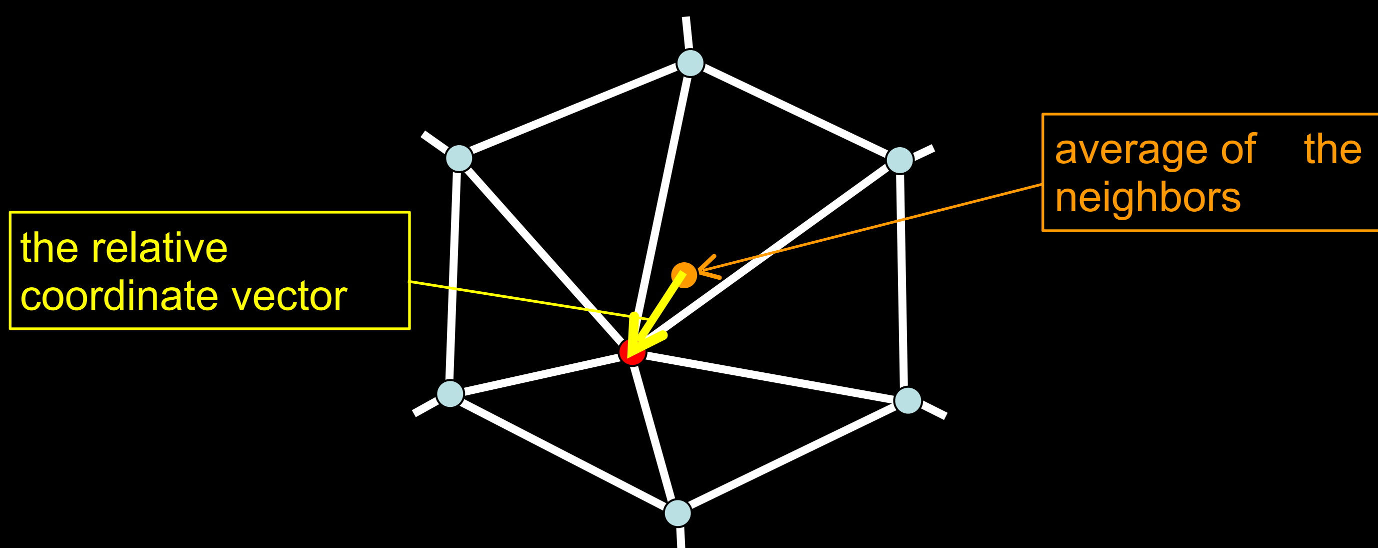
- Smooth deformation
- Smooth transition
- Preserve relative local directions of the details
- Minimal user interaction
- Interactive time response



Differential Coordinates

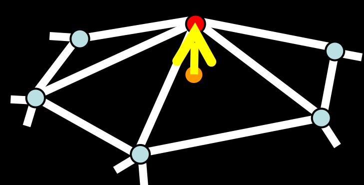
- Differential coordinates are defined for triangular mesh vertices

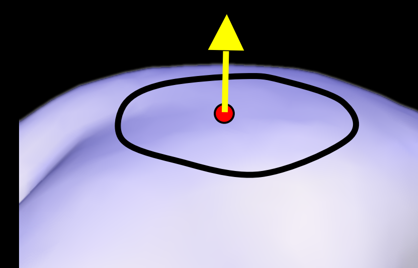
$$\boldsymbol{\delta}_{\mathrm{i}} = L(\mathbf{v}_{\mathrm{i}}) = \mathbf{v}_{\mathrm{i}} - \frac{1}{d_i} \sum_{j \in N(i)} \mathbf{v}_{\mathrm{j}}$$



Why differential coordinates?

- They represent the local detail / local shape description
  - The direction approximates the normal
  - The size approximates the mean curvature





$$\boldsymbol{\delta}_i = \frac{1}{d_i}\sum_{\mathbf{v}\in N(i)} \left(\mathbf{v}_i - \mathbf{v}\right)$$

$$\frac{1}{len(\gamma)}\int_{\mathbf{v}\in\gamma}\left(\mathbf{v}_i - \mathbf{v}\right)ds$$

$$\lim_{len(\gamma)\to 0}\frac{1}{len(\gamma)}\int_{\mathbf{v}\in\gamma}\left(\mathbf{v}_i - \mathbf{v}\right)ds = H(\mathbf{v}_i)\mathbf{n}_i$$

Laplacian reconstruction

- Transforming the mesh to the differential representation:

$$\left(\delta^{(x)}, \delta^{(y)}, \delta^{(z)}\right) = M\left(P^{(x)}, P^{(y)}, P^{(z)}\right)$$

$$\left(P^{(x)}, P^{(y)}, P^{(z)}\right) = M^{-1}\left(\delta^{(x)}, \delta^{(y)}, \delta^{(z)}\right)$$

- Note that $\mathrm{rank}(M) = n - 1$, where $n = \#V$

$$M_{ij} = \begin{cases} 1 & i = j \\ -\dfrac{1}{d_i} & j \in \{j : (j,i) \in E\} \\ 0 & otherwise \end{cases}$$

Laplacian reconstruction

- Thus for reconstructing the mesh from the Laplacian representation:

  add constraints to get full rank system and therefore unique solution, i.e. unique minimizer to the functional

$$\left\| M \cdot P^{(x)} - \delta^{(x)} \right\|^2 + \sum_{i \in I} w_i \left(p_i^{(x)} - c_i^{(x)} \right)^2$$

where I is the index set of constrained vertices, $w_i > 0$ are weights and c_i are the spatial constraints.

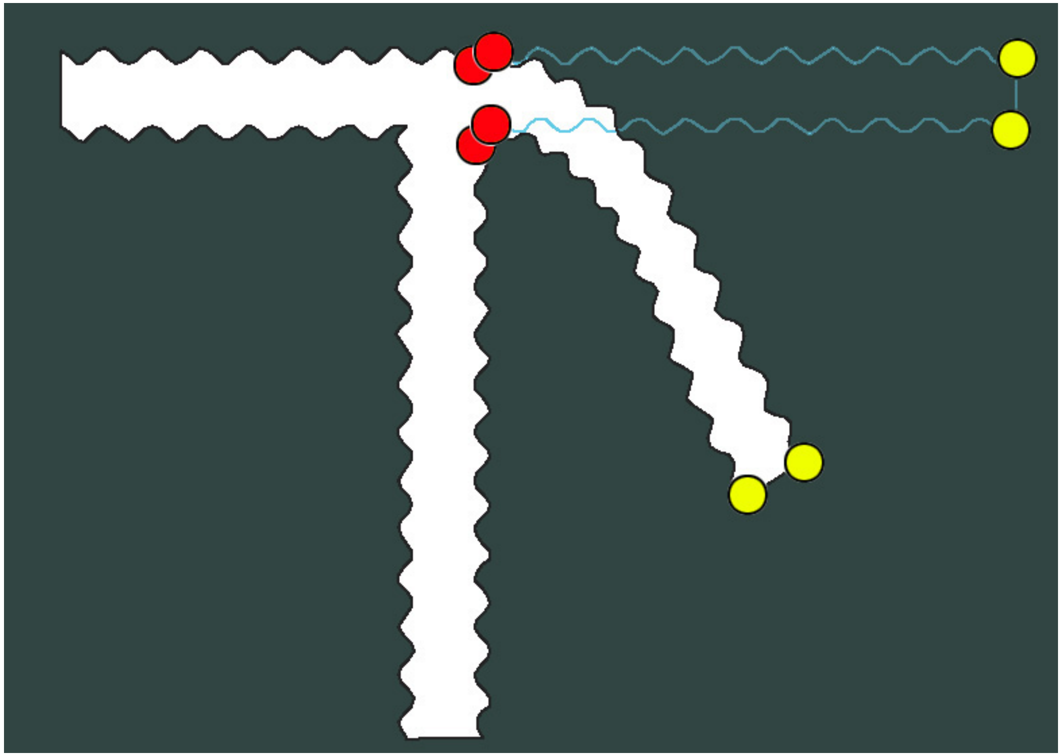
Laplacian reconstruction

The use of Laplacian (differential) representation and least squares solution forces local detail preserving

Edit a Surface While Retaining its Visual Appearance

Rotated Laplacian reconstruction

- We'd like to perform deformation which preserves the detail orientation and shape:

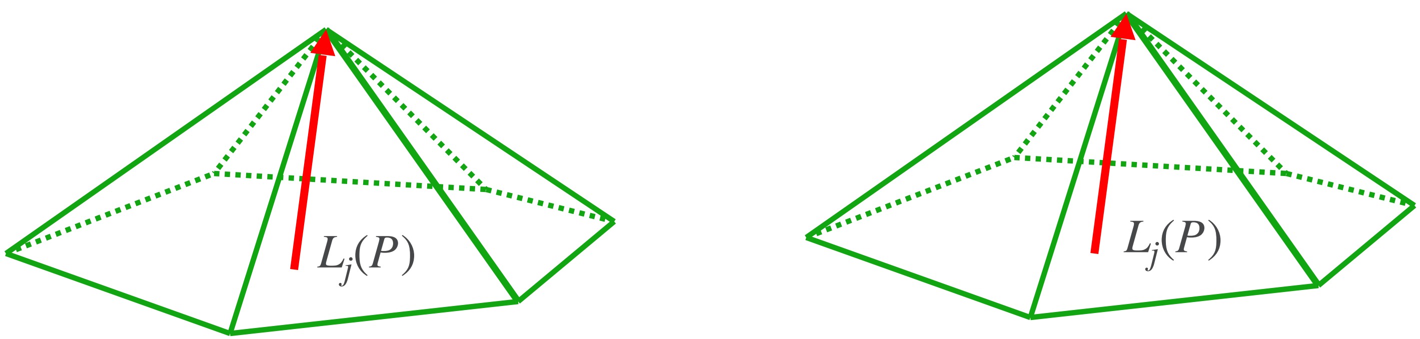


- We'd like to estimate the target shape Laplacians

Rotated Laplacian reconstruction

- The Laplacians are translation invariant:

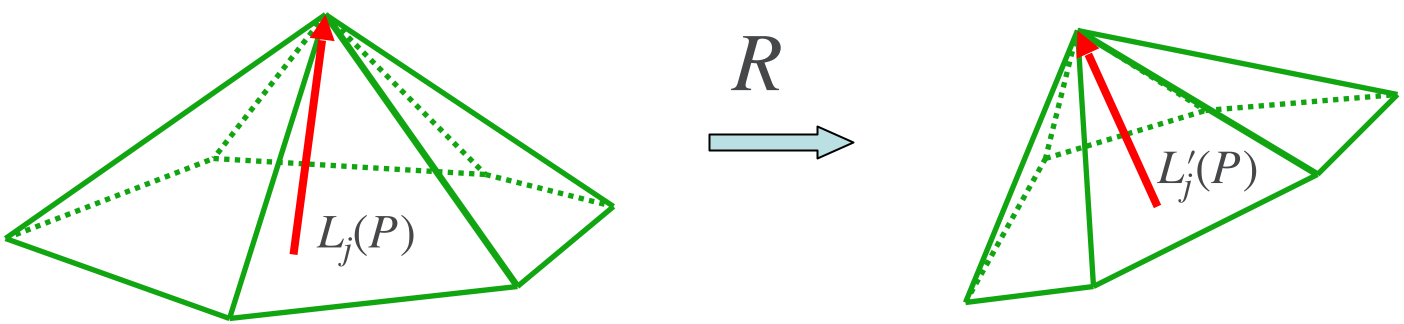
$$L_m\left(T(P)\right) = L_m(P)$$



Rotated Laplacian reconstruction

- Laplacians are not rotational invariant (they represent detail with orientation)
- Note that the Laplacian operator commute with linear rotations :

$$L_m(R(P)) = R(L_m(P))$$



Rotated Laplacian reconstruction

- Therefore we get:

$$L_j(P') = L_j(A_j(P)) =$$
$$= L_j(R_j(P)) = R_j(L_j(P))$$

- So all we need is to estimate the local rotations.

Rotated Laplacian reconstruction

- In summary we have the following steps:
-

1. Reconstruct the surface with original Laplacians:

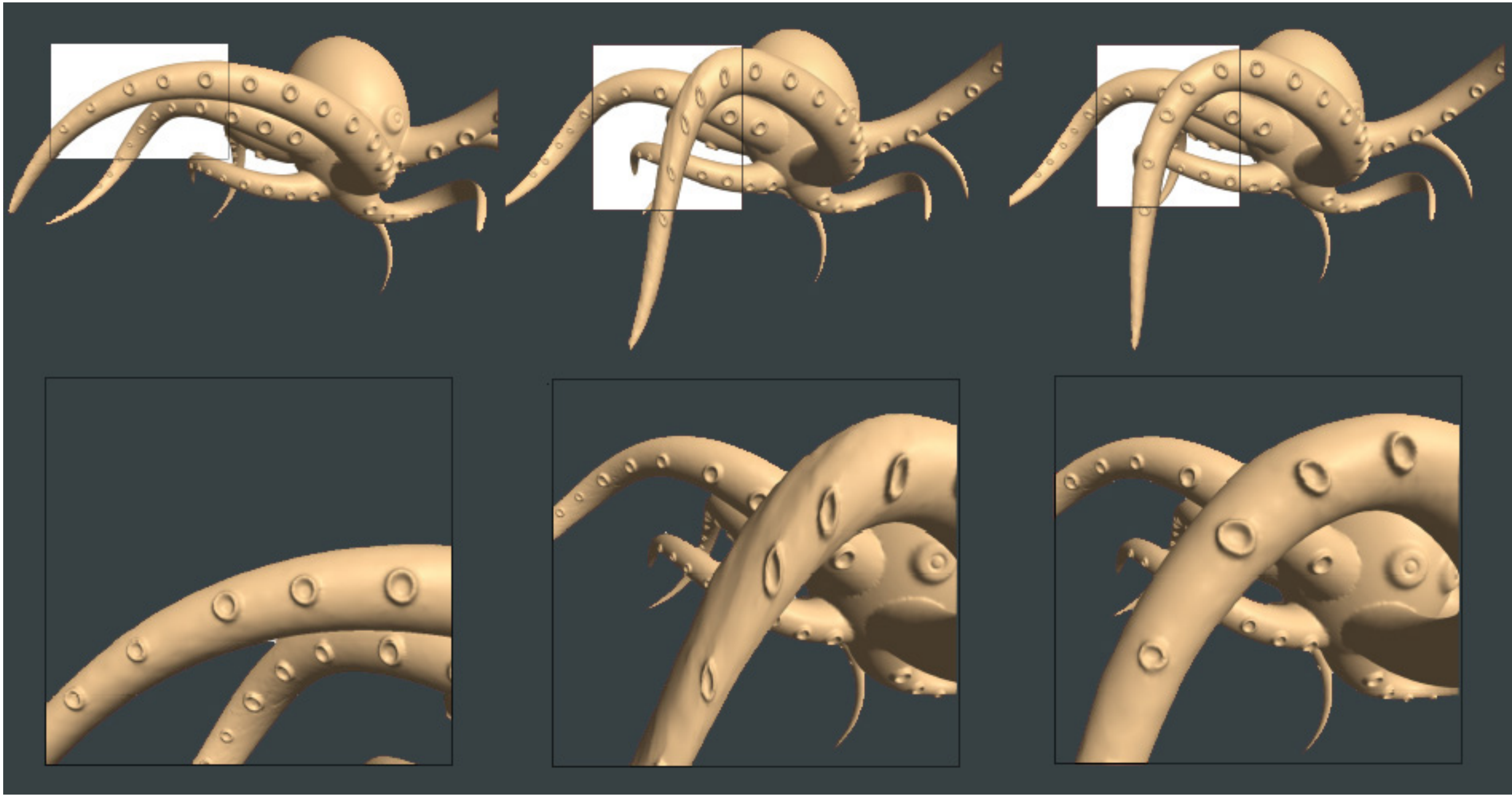
$$M^{-1}(\delta, C)$$

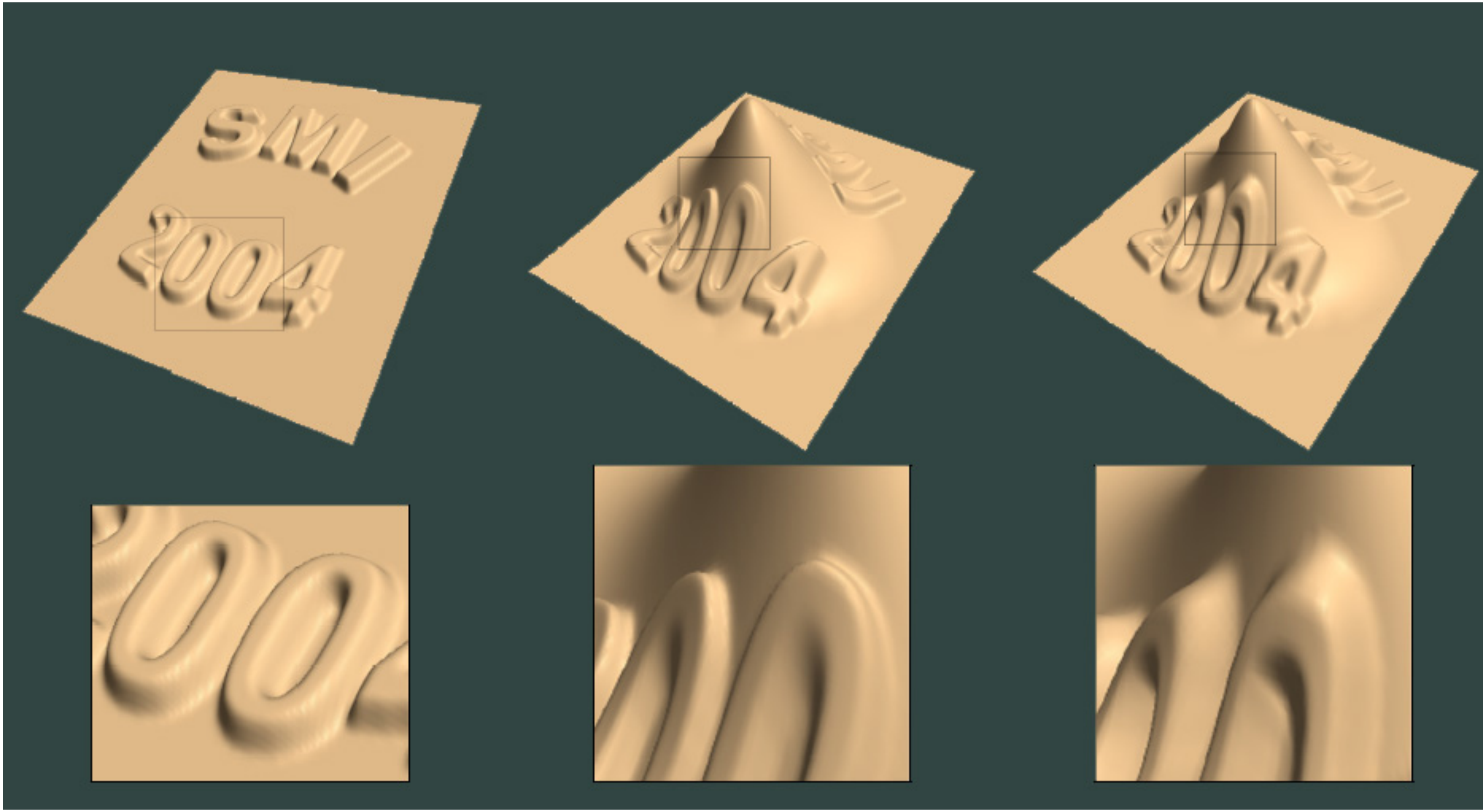
2. Approximate local rotations R_j

3. Rotate each Laplacian coordinate $L_j(P)$ by R_j

4. Reconstruct the edited surface:

$$M^{-1}\left[R_j(L_j(P)), C\right]$$

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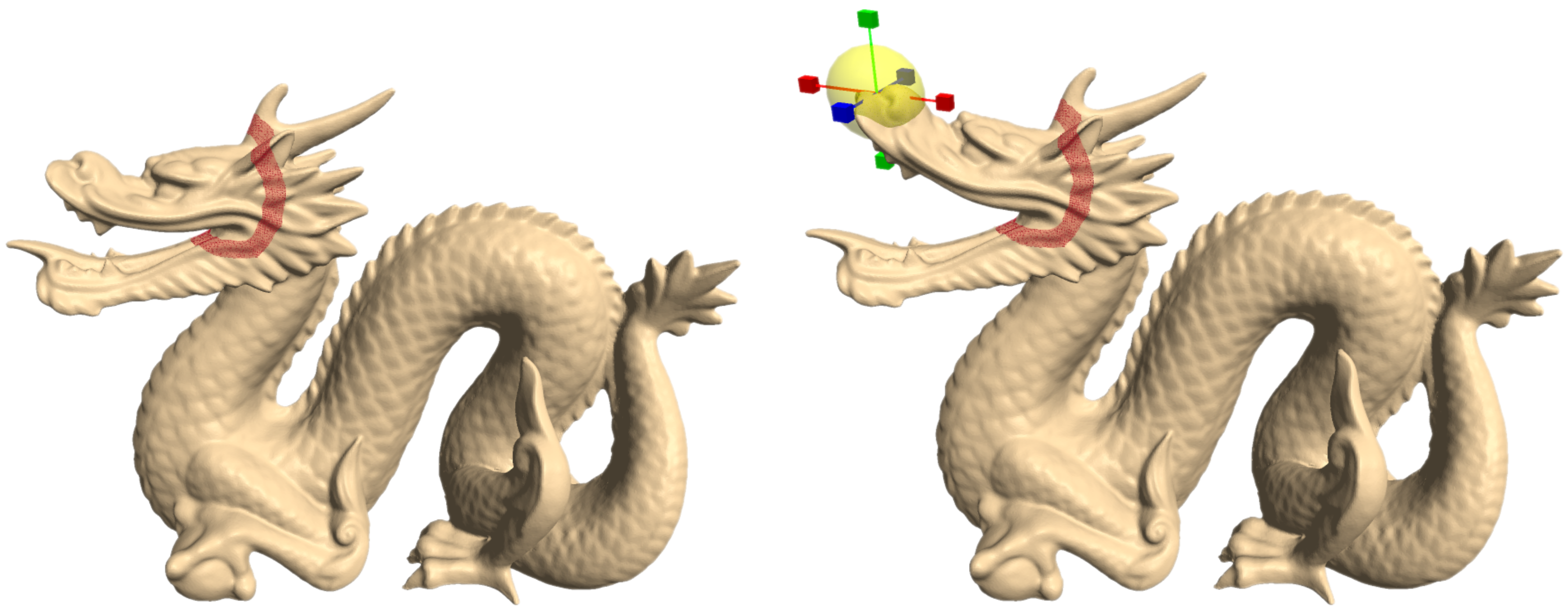
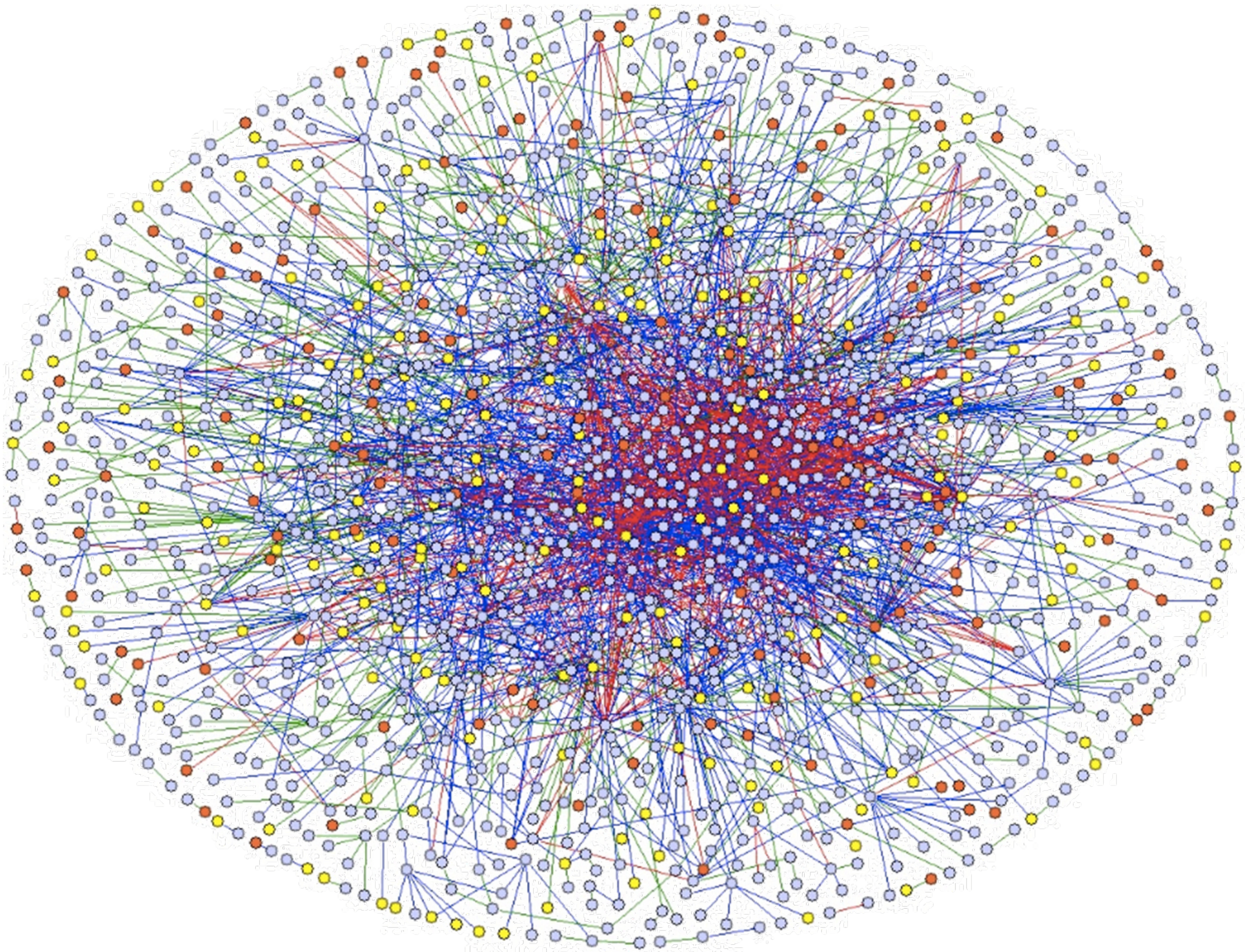


GENERAL DISCRETE GRAPH LAPLACIAN (NOTATIONS)

The Graph View of Data



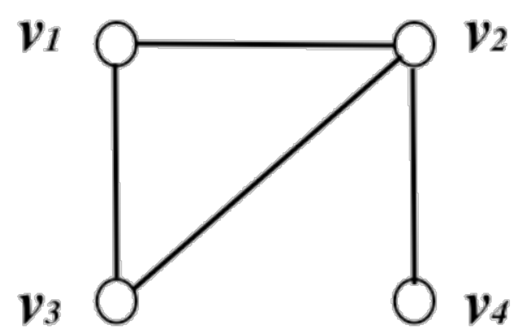
Social Networks

Adjacency Matrices

- For a graph with n vertices, the entries of the $n \times n$ adjacency matrix are defined by:

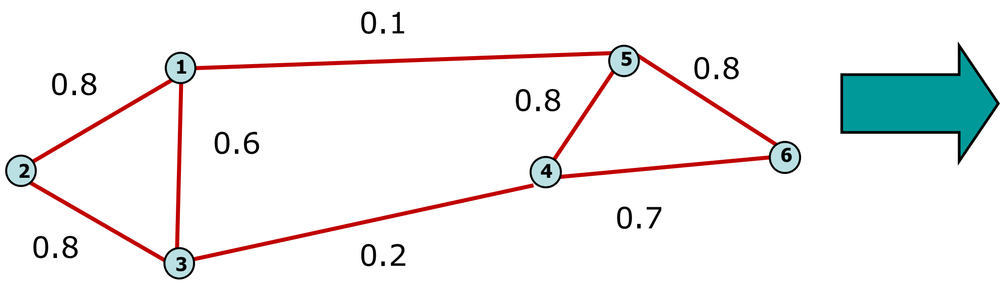
$$\mathbf{A} := \begin{cases} A_{ij} = 1 & \text{if there is an edge } e_{ij} \\ A_{ij} = 0 & \text{if there is no edge} \\ A_{ii} = 0 & \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



Weighted Matrices

- Adjacency matrix (A)
  - n x n matrix
  - $A = [w_{ij}]$: edge weight between vertex x_i and x_j

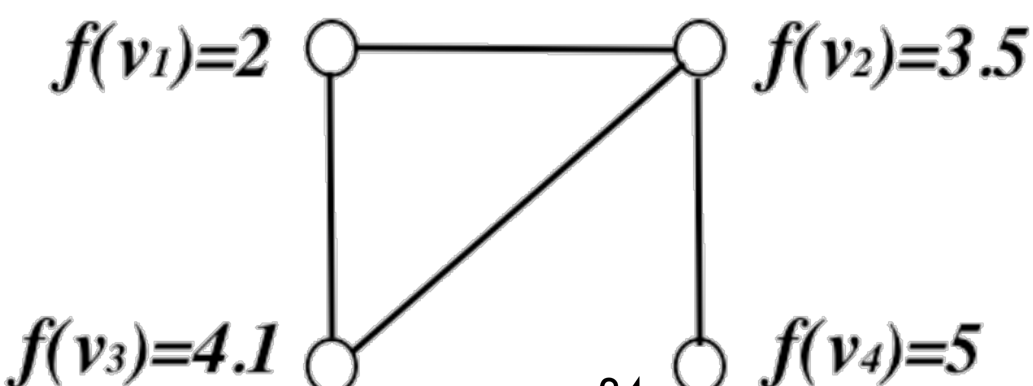


	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	0.8	0.6	0	0.1	0
x_2	0.8	0	0.8	0	0	0
x_3	0.6	0.8	0	0.2	0	0
x_4	0	0	0.2	0	0.8	0.7
x_5	0.1	0	0	0.8	0	0.8
x_6	0	0	0	0.7	0.8	0

- Important properties:
  - Symmetric matrix
  - $\Rightarrow$ Eigenvalues are <u>real</u>
  - $\Rightarrow$ Eigenvector could span <u>orthogonal base</u>

Functions on Graphs

- We consider real-valued functions on the set of the graph's vertices, $\boldsymbol{f} : \mathcal{V} \longrightarrow \mathbb{R}$. Such a function assigns a real number to each graph node.
- $\boldsymbol{f}$ is a vector indexed by the graph's vertices, hence $\boldsymbol{f} \in \mathbb{R}^n$.
- Notation: $\boldsymbol{f} = (f(v_1), \ldots, f(v_n)) = (f(1), \ldots, f(n))$.
- The eigenvectors of the adjacency matrix, $\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}$, can be viewed as *eigenfunctions*.



Operators and Quadratic Forms

- The adjacency matrix can be viewed as an operator

$$\boldsymbol{g} = \mathbf{A}\boldsymbol{f}; g(i) = \sum_{i \sim j} f(j)$$

- It can also be viewed as a quadratic form:

$$\boldsymbol{f}^\top \mathbf{A}\boldsymbol{f} = \sum_{e_{ij}} f(i)f(j)$$

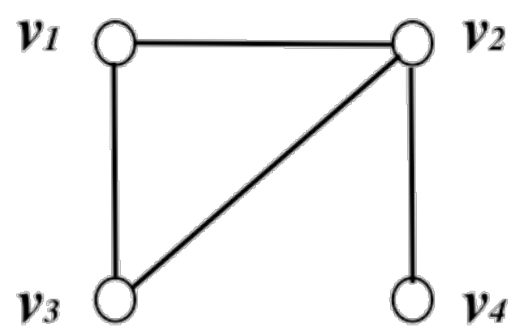
Graph (Unnormalized) Laplacian

- $\mathbf{L} = \nabla^\top \nabla$
- $(\mathbf{L}\boldsymbol{f})(v_i) = \sum_{v_j \sim v_i} (f(v_i) - f(v_j))$
- Connection between the Laplacian and the adjacency matrices:

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

- The degree matrix: $\mathbf{D} := D_{ii} = d(v_i)$.

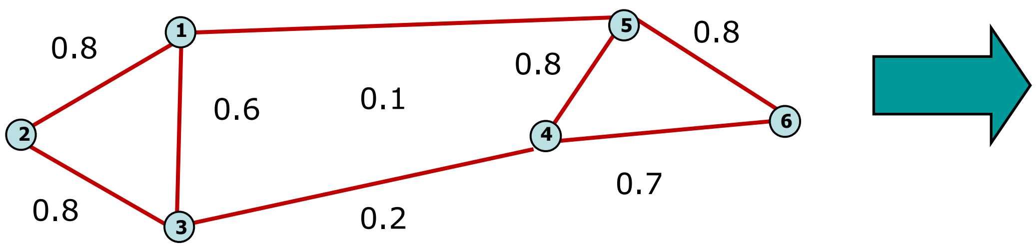
$$\mathbf{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$



Laplacian Matrix

- **Laplacian matrix (L)**
  - $n \times n$ symmetric matrix

$$L = D - A$$



	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1.5	-0.8	-0.6	0	-0.1	0
x_2	-0.8	1.6	-0.8	0	0	0
x_3	-0.6	-0.8	1.6	-0.2	0	0
x_4	0	0	-0.2	1.7	-0.8	-0.7
x_5	-0.1	0	0	0.8-	1.7	-0.8
x_6	0	0	0	-0.7	-0.8	1.5

- Important properties:
  - Eigenvalues are non-negative real numbers (Gershgorin circle theorem)
  - Eigenvectors are real and orthogonal
  - Eigenvalues and eigenvectors provide an insight into the connectivity of the graph…

Laplacian Defines Natural Quadratic Form of Graphs

$$x^T L x = \sum_{(i,j)\in E} (x(i) - x(j))^2$$

$L = D - A$ where D is diagonal matrix of degrees

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Undirected Weighted Graphs

- We consider *undirected weighted graphs*: Each edge e_{ij} is weighted by $w_{ij} > 0$.
- The Laplacian as an operator:

$$(\mathbf{L}\boldsymbol{f})(v_i) = \sum_{v_j \sim v_i} w_{ij}(f(v_i) - f(v_j))$$

- As a quadratic form:

$$\boldsymbol{f}^\top \mathbf{L}\boldsymbol{f} = \frac{1}{2}\sum_{e_{ij}} w_{ij}(f(v_i) - f(v_j))^2$$

- $\mathbf{L}$ is symmetric and positive semi-definite.
- $\mathbf{L}$ has n non-negative, real-valued eigenvalues: $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

GENERAL DISCRETE GRAPH LAPLACIAN (SOME PROPERTIES)

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Connected Graph Laplacians

- $\mathbf{L}\boldsymbol{u} = \lambda\boldsymbol{u}$.
- $\mathbf{L}\mathbf{1}_n = \mathbf{0}$, $\lambda_1 = 0$ is the smallest eigenvalue.
- The *one* vector: $\mathbf{1}_n = (1 \dots 1)^\top$.
- $0 = \boldsymbol{u}^\top \mathbf{L}\boldsymbol{u} = \sum_{i,j=1}^{n} w_{ij}(u(i) - u(j))^2$.
- If any two vertices are connected by a path, then $\boldsymbol{u} = (u(1), \dots, u(n))$ needs to be constant at all vertices such that the quadratic form vanishes. Therefore, a graph with one connected component has the constant vector $\boldsymbol{u}_1 = \mathbf{1}_n$ as the only eigenvector with eigenvalue 0.

A Graph with k Connected Components

- Each connected component has an associated Laplacian. Therefore, we can write matrix $\mathbf{L}$ as a *block diagonal matrix*:

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & & \\ & \ddots & \\ & & \mathbf{L}_k \end{bmatrix}$$

- The spectrum of $\mathbf{L}$ is given by the union of the spectra of $\mathbf{L}_i$.
- Each block corresponds to a connected component, hence each matrix $\mathbf{L}_i$ has an eigenvalue 0 with multiplicity 1.
- The spectrum of $\mathbf{L}$ is given by the union of the spectra of $\mathbf{L}_i$.
- The eigenvalue $\lambda_1 = 0$ has multiplicity k.

The Eigenspace of λ = 0

- The eigenspace corresponding to $\lambda_1 = \ldots = \lambda_k = 0$ is spanned by the k mutually orthogonal vectors:

$$\boldsymbol{u}_1 = \mathbf{1}_{L_1}$$
$$\ldots$$
$$\boldsymbol{u}_k = \mathbf{1}_{L_k}$$

- with $\mathbf{1}_{L_i} = (0000111110000)^\top \in \mathbb{R}^n$
- These vectors are the *indicator vectors* of the graph's connected components.
- Notice that $\mathbf{1}_{L_1} + \ldots + \mathbf{1}_{L_k} = \mathbf{1}_n$

The Fiedler Vector

- The first non-null eigenvalue λ_{k+1} is called the Fiedler value.
- The corresponding eigenvector $\boldsymbol{u}_{k+1}$ is called the Fiedler vector.
- The multiplicity of the Fiedler eigenvalue is always equal to 1.
- The Fiedler value is the *algebraic connectivity of a graph*, the further from 0, the more connected.
- The Fidler vector has been extensively used for *spectral bi-partioning*
- Theoretical results are summarized in Spielman & Teng 2007: `http://cs-www.cs.yale.edu/homes/spielman/`

Laplacian Eigenvectors for Connected Graphs

- $\boldsymbol{u}_1 = \mathbf{1}_n, \mathbf{L}\mathbf{1}_n = \mathbf{0}$.
- $\boldsymbol{u}_2$ is the *the Fiedler vector* with multiplicity 1.
- The eigenvectors form an orthonormal basis: $\boldsymbol{u}_i^\top \boldsymbol{u}_j = \delta_{ij}$.
- For any eigenvector $\boldsymbol{u}_i = (\boldsymbol{u}_i(v_1) \ldots \boldsymbol{u}_i(v_n))^\top$, $2 \leq i \leq n$:

$$\boldsymbol{u}_i^\top \mathbf{1}_n = 0$$

- Hence the components of $\boldsymbol{u}_i$, $2 \leq i \leq n$ satisfy:

$$\sum_{j=1}^{n} \boldsymbol{u}_i(v_j) = 0$$

λ_2 = algebraic connectivity, monotone under graph inclusion

- Each component is bounded by:

$$-1 < \boldsymbol{u}_i(v_j) < 1$$

Some Special Graphs

- The complete graph on n vertices, K_n, which has edge set $\{(u, v) : u \neq v\}$.
- The star graph on n vertices, S_n, which has edge set $\{(1, u) : 2 \leq u \leq n\}$.
- The hypercube

The **hypercube** on 2^k vertices. The vertices are elements of $\{0,1\}^k$. Edges exist between vertices that differ in only one coordinate.

Complete Graph

Lemma 2.5.1. *The Laplacian of K_n has eigenvalue* 0 *with multiplicity* 1 *and* n *with multiplicity* $n-1$.

Proof. To compute the non-zero eigenvalues, let $\boldsymbol{\psi}$ be any non-zero vector orthogonal to the all-1s vector, so

$$\sum_u \boldsymbol{\psi}(u) = 0. \tag{2.6}$$

We now compute the first coordinate of $\boldsymbol{L}_{K_n}\boldsymbol{\psi}$. Using (2.3), we find

$$(\boldsymbol{L}_{K_n}\boldsymbol{\psi})(1) = \sum_{v \geq 2} (\boldsymbol{\psi}(1) - \boldsymbol{\psi}(v)) = (n-1)\boldsymbol{\psi}(1) - \sum_{v=2}^{n} \boldsymbol{\psi}(v) = n\boldsymbol{\psi}(1), \quad \text{by (2.6)}.$$

As the choice of coordinate was arbitrary, we have $\boldsymbol{L}\boldsymbol{\psi} = n\boldsymbol{\psi}$. So, every vector orthogonal to the all-1s vector is an eigenvector of eigenvalue n. □

Alternative approach. Observe that $\boldsymbol{L}_{K_n} = n\boldsymbol{I} - \mathbf{1}\mathbf{1}^T$. □

Star Graph

Lemma 2.5.2. *Let $G = (V, E)$ be a graph, and let v and w be vertices of degree one that are both connected to another vertex z. Then, the vector $\boldsymbol{\psi} = \boldsymbol{\delta}_v - \boldsymbol{\delta}_w$ is an eigenvector of $\boldsymbol{L}_G$ of eigenvalue 1.*

Proof. Just multiply $\boldsymbol{L}_G$ by $\boldsymbol{\psi}$, and check vertex-by-vertex that it equals $\boldsymbol{\psi}$. $\square$

As eigenvectors of different eigenvalues are orthogonal, this implies that $\boldsymbol{\psi}(u) = \boldsymbol{\psi}(v)$ for every eigenvector with eigenvalue different from 1.

Lemma 2.5.3. *The graph S_n has eigenvalue 0 with multiplicity 1, eigenvalue 1 with multiplicity $n - 2$, and eigenvalue n with multiplicity 1.*

Hypercube Graph

- Exercise