UC San Diego

Laplacian Basics

(Smoothing, Cotangent Laplacian)

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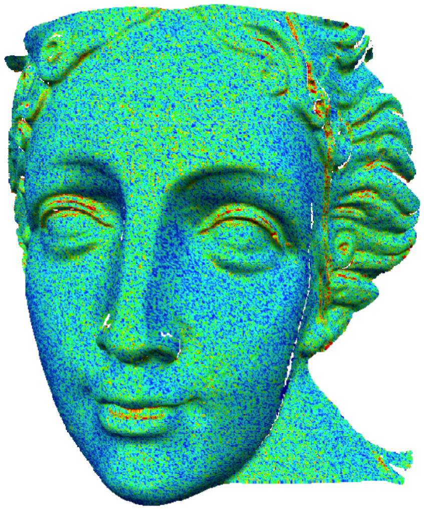
MESH SMOOTHING
(AKA DENOISING, FILTERING, FAIRING)

UC San Diego

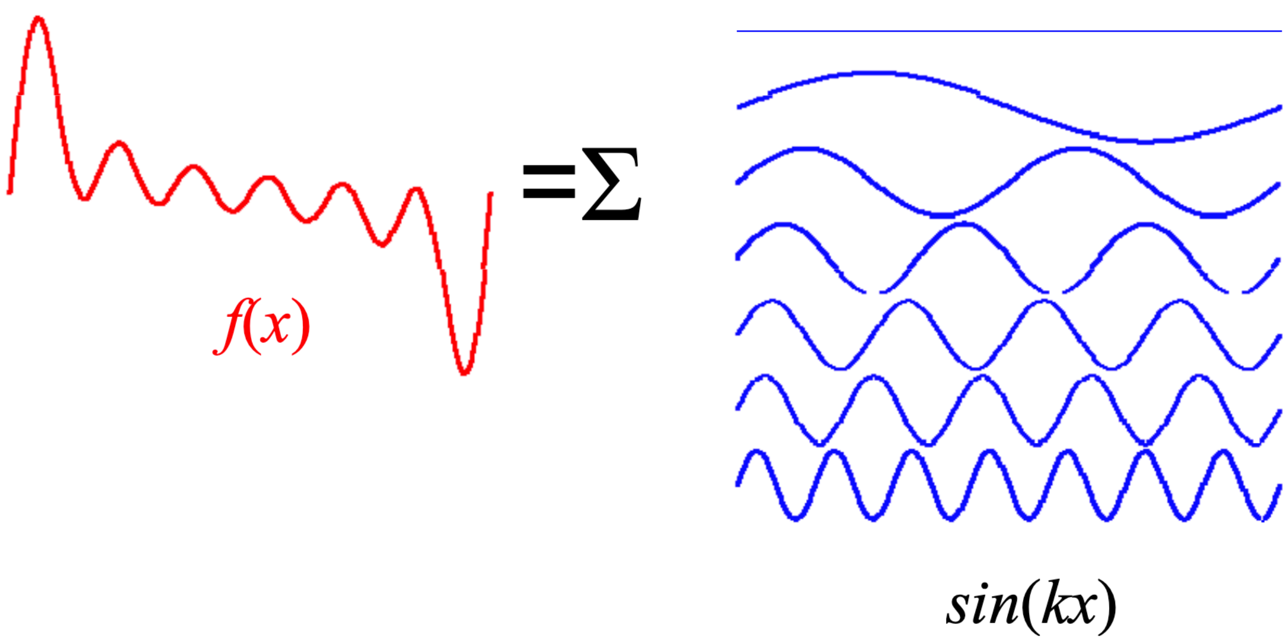
Mesh Smoothing

- Input: Noisy mesh (scanned or other)
- Output: Smooth mesh
- How: Filter out high frequency noise



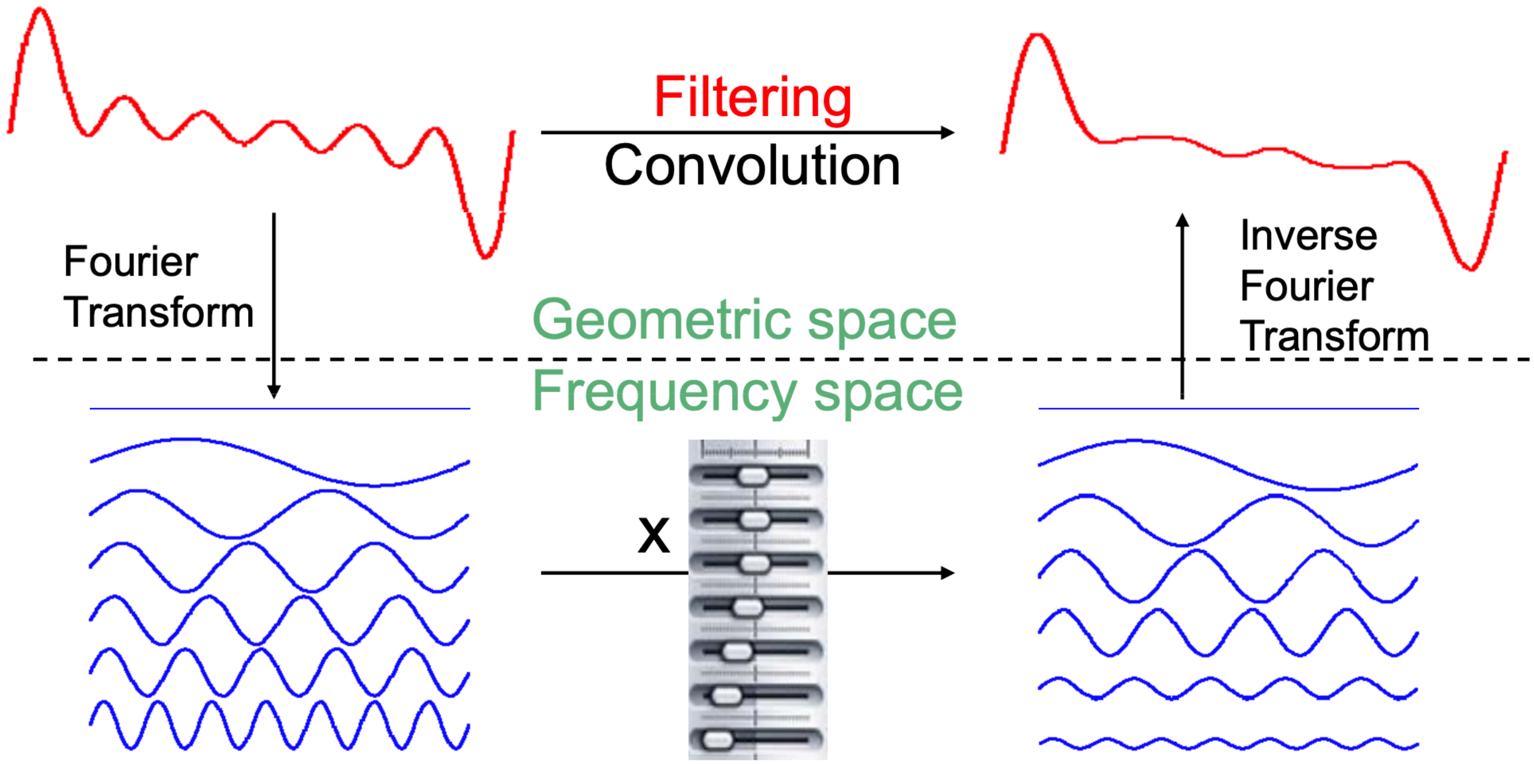
Smoothing by Filtering

Fourier Transform

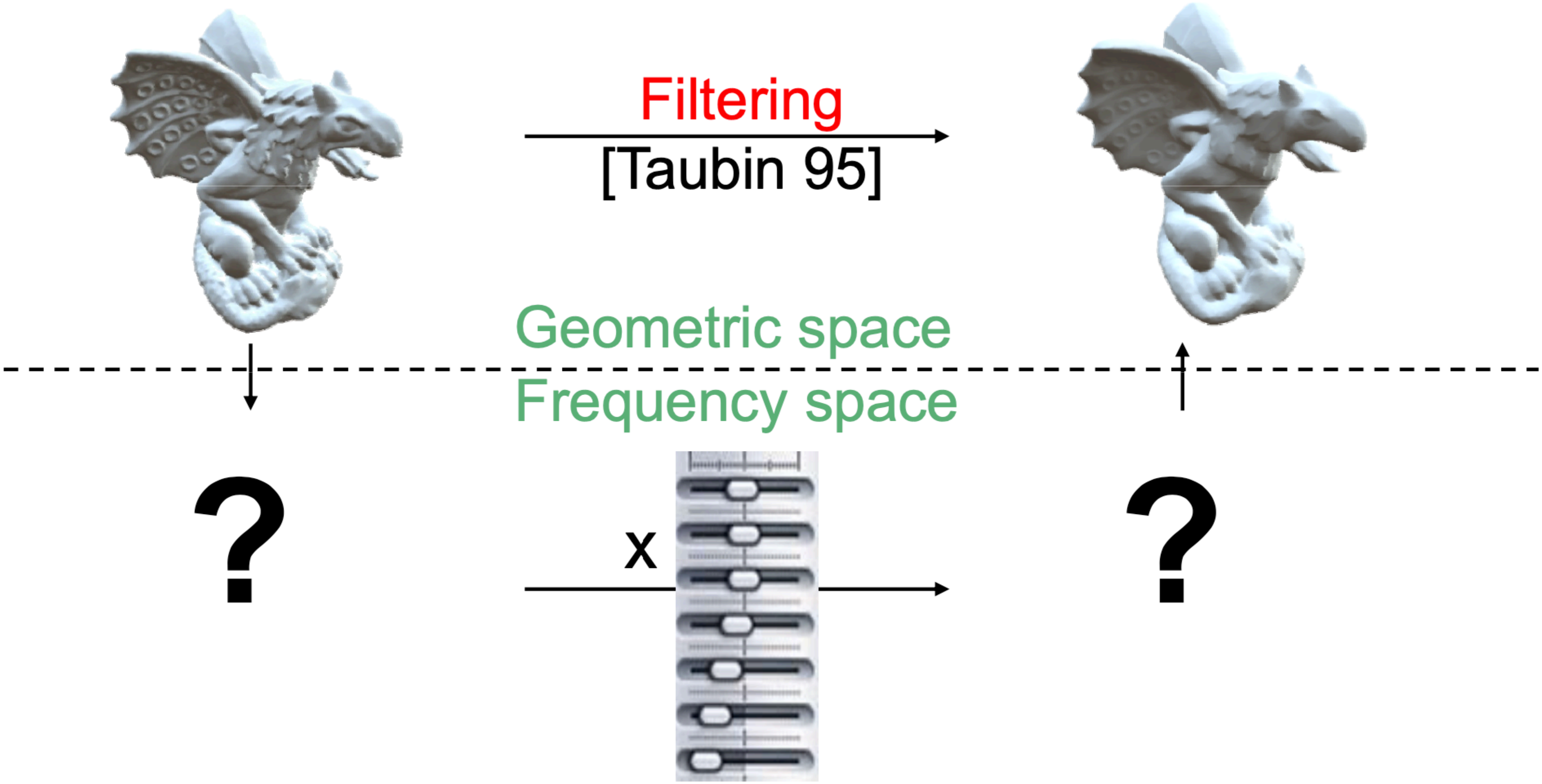


Smoothing by Filtering

Fourier Transform

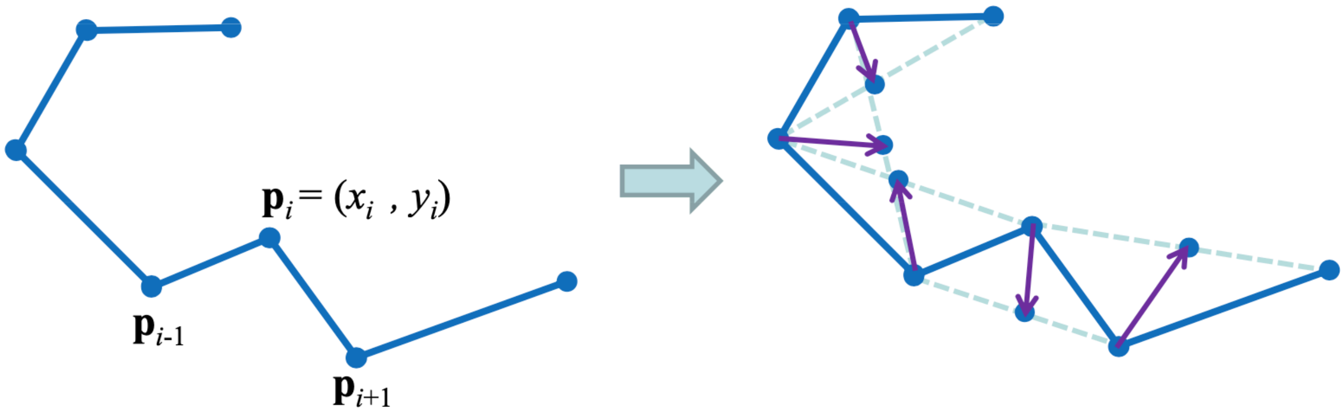


Filtering on a Mesh



Laplacian Smoothing

- An easier problem: How to smooth a curve?

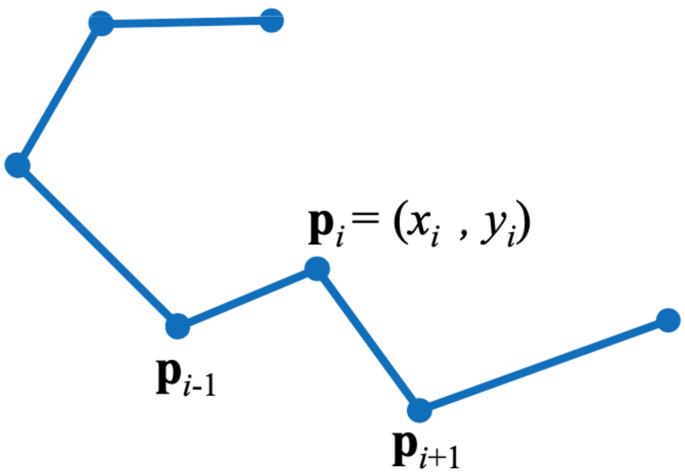


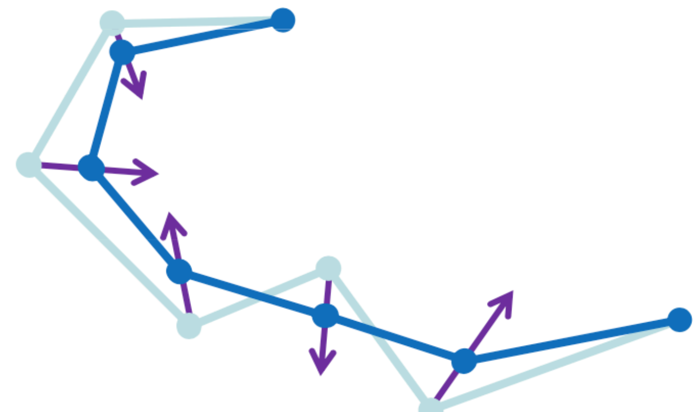
$(\mathbf{p}_{i\text{-}1} + \mathbf{p}_{i+1})/2 - \mathbf{p}_i$

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

Laplacian Smoothing

- An easier problem: How to smooth a curve?





$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \frac{1}{2} L(\mathbf{p}_i)$$

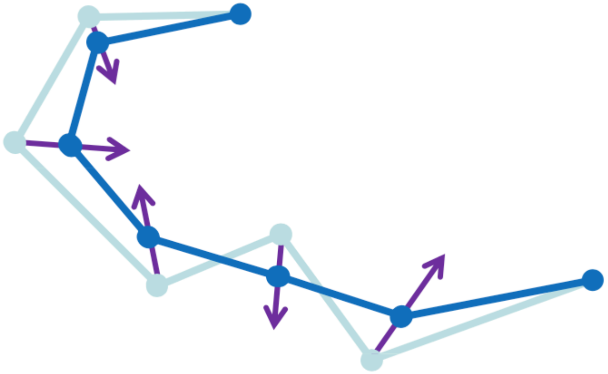
Finite difference discretization of second derivative
= Laplace operator in one dimension

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

Laplacian Smoothing

<u>Algorithm:</u>
Repeat for *m* iterations (for non boundary points):

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda L(\mathbf{p}_i)$$



For which λ?
$0 < \lambda < 1$

Closed curve converges to?
Single point

Spectral Analysis

- Closed curve

Re-write $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda L\left(\mathbf{p}_i^{(t)}\right)$

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

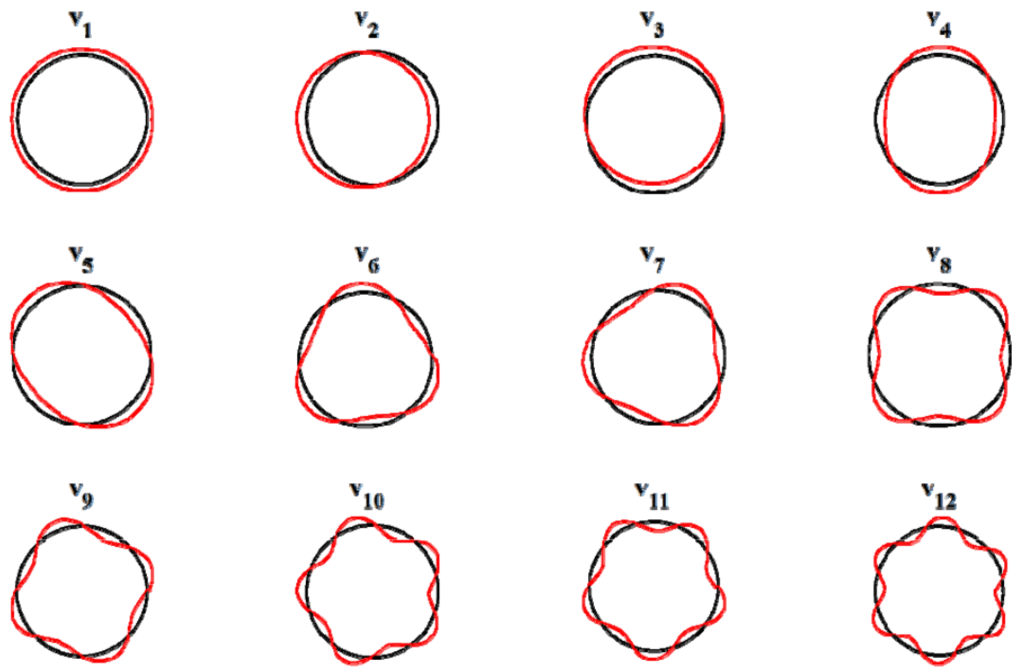
in matrix notation: $\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L} \mathbf{P}^{(t)}$

$$\mathbf{P} = \begin{pmatrix} x_1 & y_2 \\ \dots & \dots \\ x_n & y_n \end{pmatrix} \in \mathbb{R}^{n \times 2} \qquad \mathbf{L} = \frac{1}{2} \begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & & \dots & & & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

The Eigenvectors of L

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^T$$

$$\mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \cdots & \\ & & & k_n \end{pmatrix}$$



$$0 \leq \lambda(L) \leq 2$$

Lemma 5.3.1. *The Laplacian of* R_n *has eigenvectors*

$$\boldsymbol{x}_k(u) = \cos(2\pi ku/n), \text{ and}$$
$$\boldsymbol{y}_k(u) = \sin(2\pi ku/n),$$

for $0 \leq k \leq n/2$*, ignoring* $\boldsymbol{y}_0$ *which is the all-zero vector, and for even* n *ignoring* $\boldsymbol{y}_{n/2}$ *for the same reason. Eigenvectors* $\boldsymbol{x}_k$ *and* $\boldsymbol{y}_k$ *have eigenvalue* $2 - 2\cos(2\pi k/n)$*.*

http://www.cs.yale.edu/homes/spielman/561/2012/lect05-12.pdf

Spectral Analysis

Then: $\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L} \mathbf{P}^{(t)} = (\mathbf{I} - \lambda \mathbf{L}) \mathbf{P}^{(t)}$

After m iterations: $\mathbf{P}^{(m)} = (\mathbf{I} - \lambda \mathbf{L})^{m} \mathbf{P}^{(0)}$

Can be described using eigen-decomposition of $\mathbf{L}$

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^{T}$$

$$\mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \cdots & \\ & & & k_n \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{P}^{(m)} = \mathbf{V} (\mathbf{I} - \lambda \mathbf{D})^{m} \mathbf{V}^{T} \mathbf{P}^{(0)}$$

Filtering high frequencies

Laplacian Smoothing on Meshes

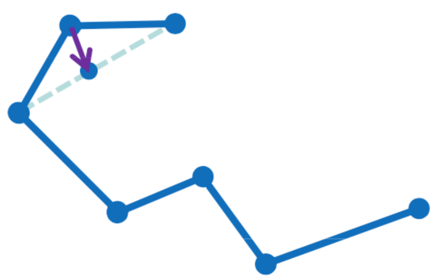
Same as for curves:

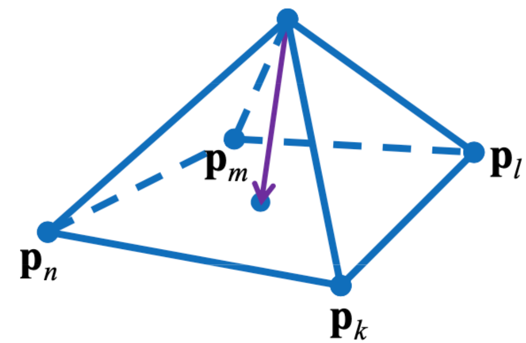
$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$$

$N_i = \{k, l, m, n\}$

$\mathbf{p}_i = (x_i, y_i, z_i)$

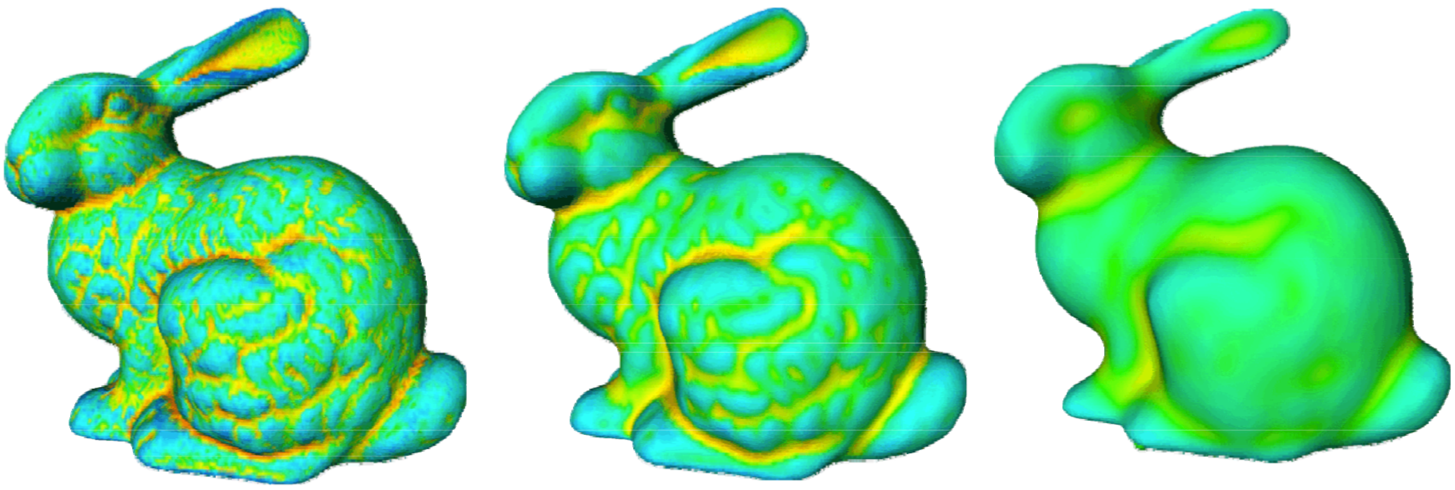
What is $\Delta \mathbf{p}_i$?





$$\frac{1}{2}\left(\mathbf{p}_{i+1} + \mathbf{p}_{i-1}\right) - \mathbf{p}_i \qquad \frac{1}{|N_i|}\left(\sum_{j \in N_i} \mathbf{p}_j\right) - \mathbf{p}_i$$

Laplacian Smoothing on Meshes



0 Iterations

5 Iterations

20 Iterations

Laplacian Smoothing

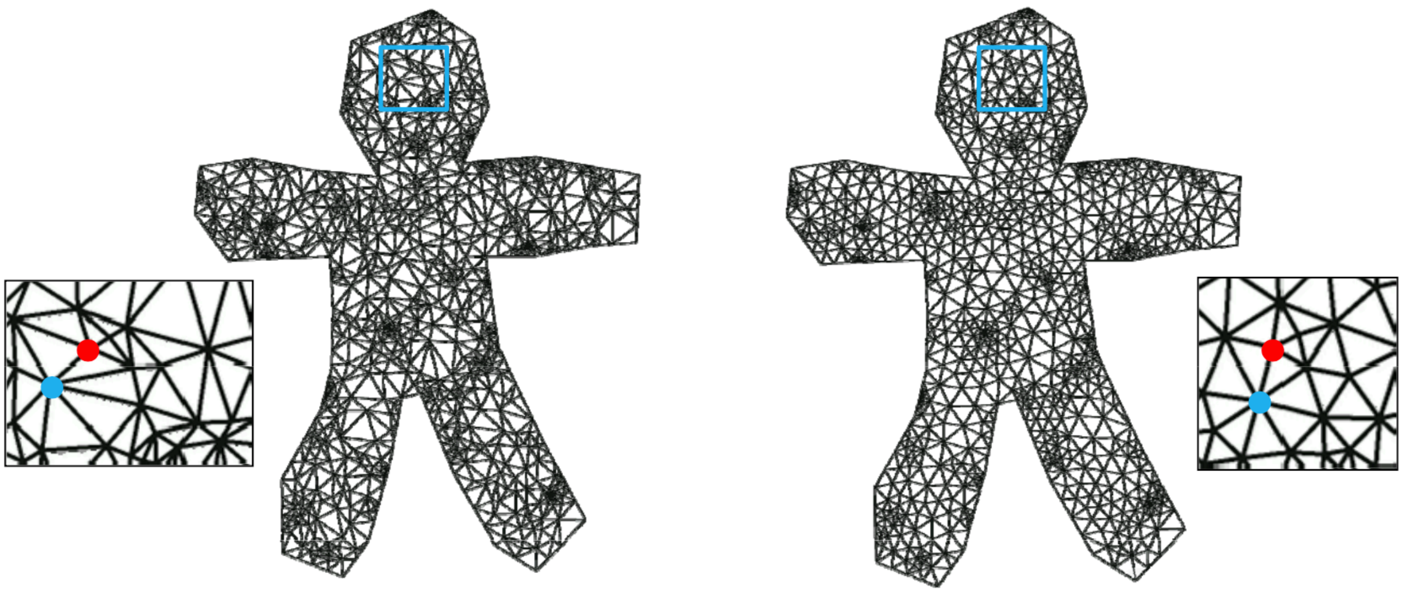
$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$$

$\Delta \mathbf{p}_i$ = mean curvature normal

mean curvature flow

Laplace Operator Discretization

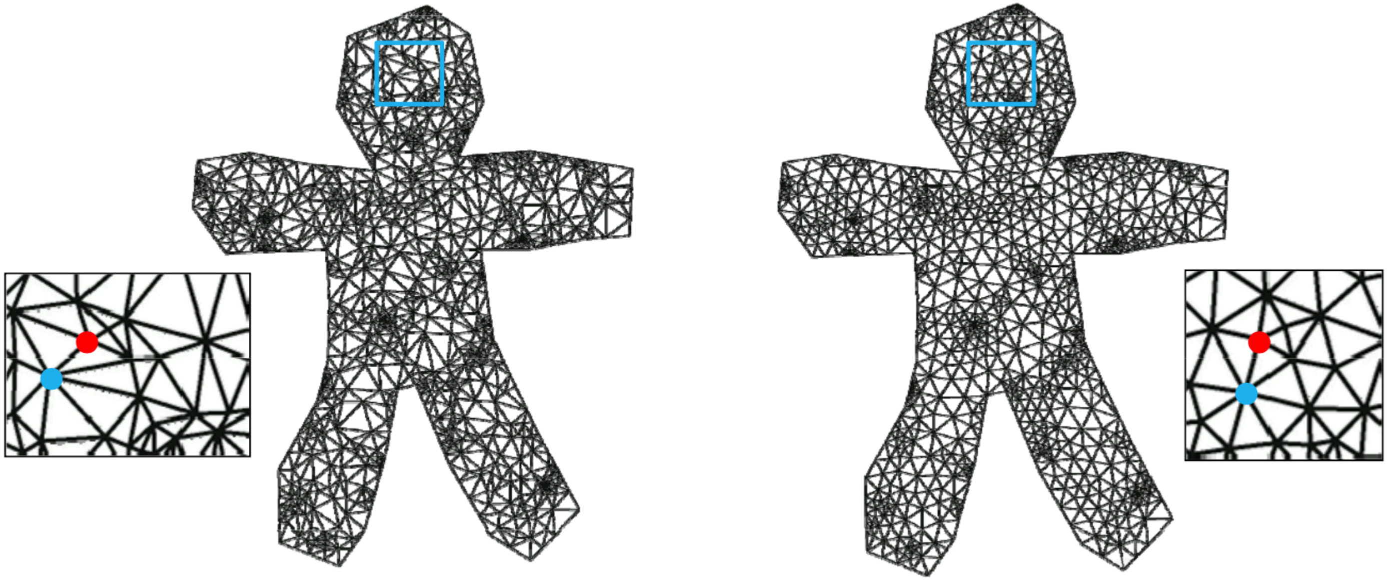
- Sanity check — what should happen if the mesh lies in the plane: $p_i = (x_i, y_i, 0)$?



0 Iterations 5 Iterations

Laplace Operator Discretization

Not good – A flat mesh is smooth, should stay the same after smoothing

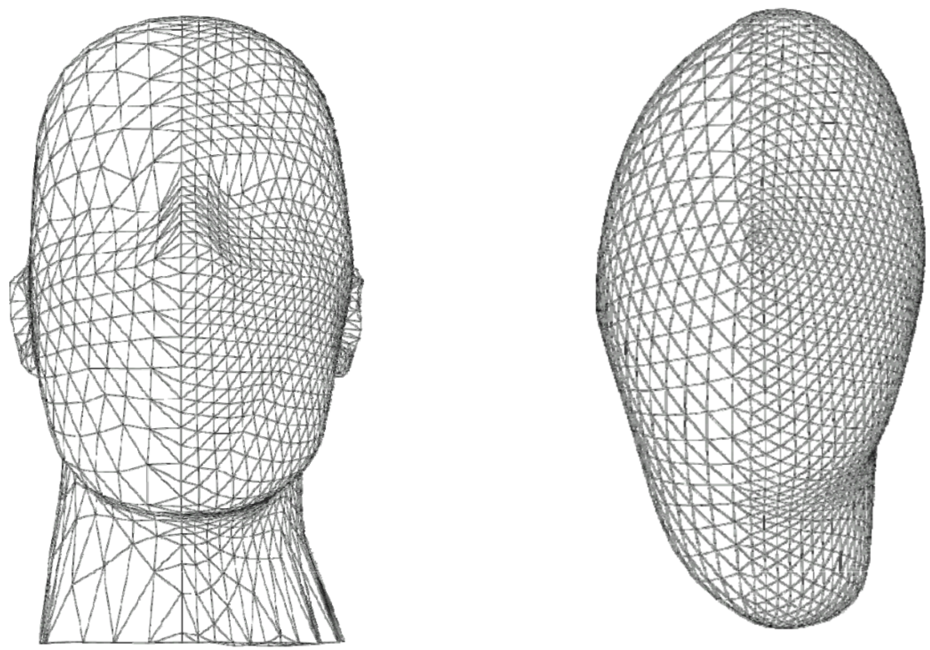


0 Iterations

5 Iterations

Laplace Operator Discretization

Not good – The result should not depend on triangle sizes



From Desbrun et al., Siggraph 1999

What Went Wrong?

Back to curves:

$$\boxed{\frac{1}{2}}\left(\mathbf{p}_{i+1}+\mathbf{p}_{i-1}\right)-\mathbf{p}_i$$

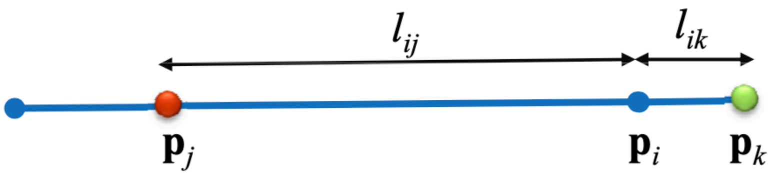
Same weight for both neighbors, although one is closer

The Solution (1D)

Use a weighted average to define Δ

Which weights?

$$w_{ij} = \frac{1}{l_{ij}} \qquad w_{ik} = \frac{1}{l_{ik}}$$



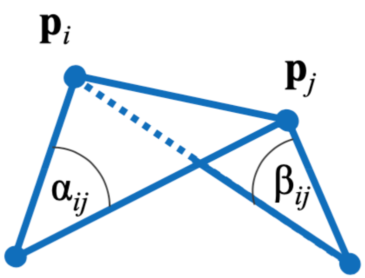
$$L(\mathbf{p}_i) = \frac{w_{ij}\mathbf{p}_j + w_{ik}\mathbf{p}_k}{w_{ij} + w_{ik}} - \mathbf{p}_i$$

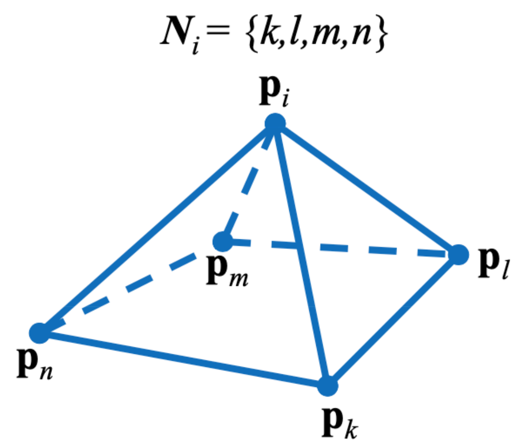
Straight curves will be invariant to smoothing

Solution (2D)

Use a weighted average to define Δ

Which weights?



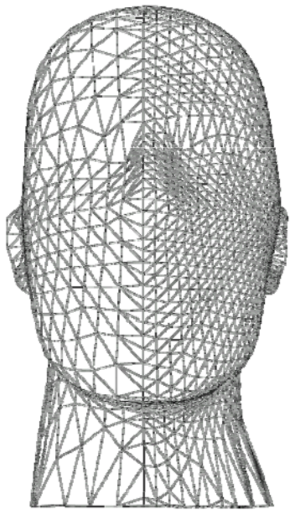


$$w_{ij} = \frac{h_{ij}^1 + h_{ij}^2}{l_{ij}} = \frac{1}{2}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right)$$

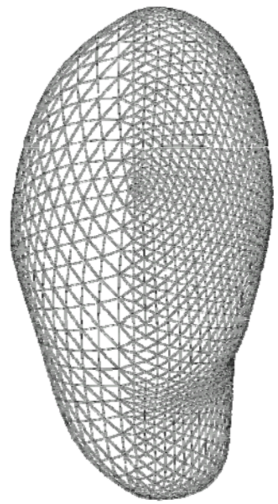
$$L(\mathbf{p}_i) = \frac{1}{\sum_{j\in N_i} w_{ij}}\left(\sum_{j\in N_i} w_{ij}\mathbf{p}_j\right) - \mathbf{p}_i$$

Planar meshes will be invariant to smoothing

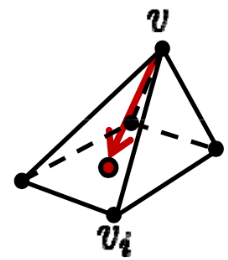
Smoothing with the Cotangent Laplacian



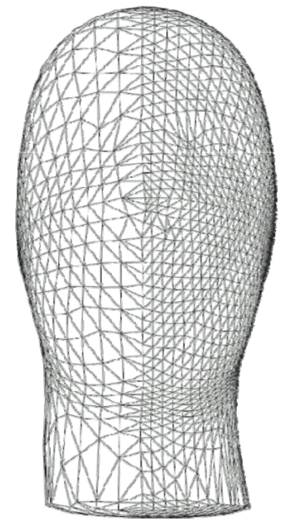
original



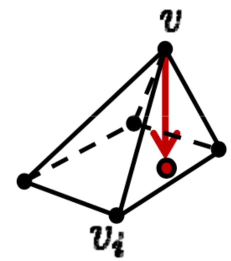
Uniform weights



normal and tangential movement



Cotangent weights

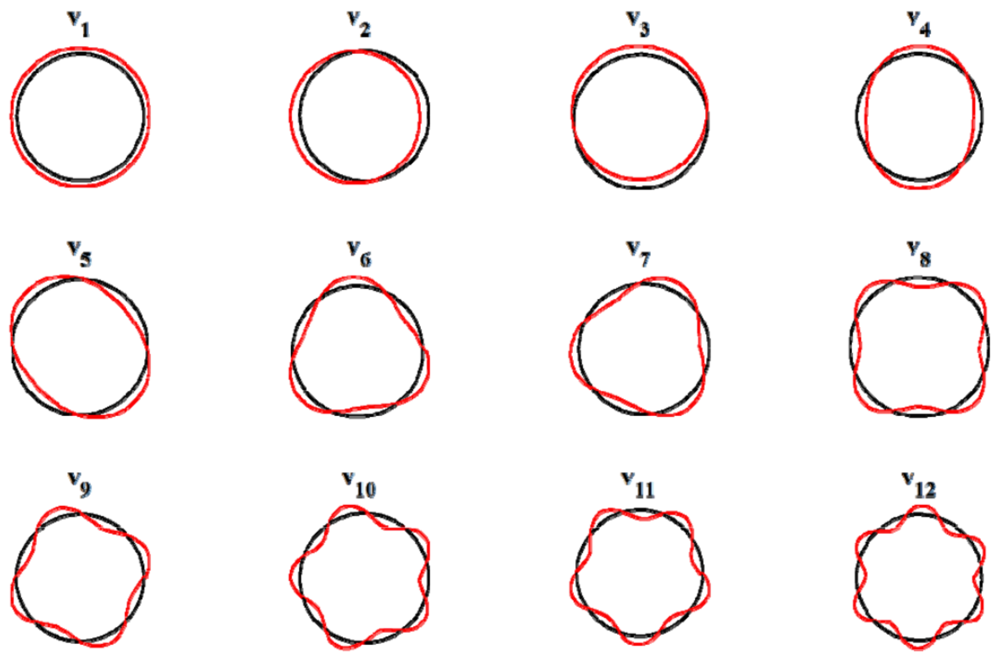


normal movement

From Desbrun et al., Siggraph 1999

The Eigenvectors of L

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^T \qquad \mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \dots & \\ & & & k_n \end{pmatrix}$$



Spectral Analysis

- Cotangent Laplacian

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^T \qquad \mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \dots & \\ & & & k_n \end{pmatrix}$$

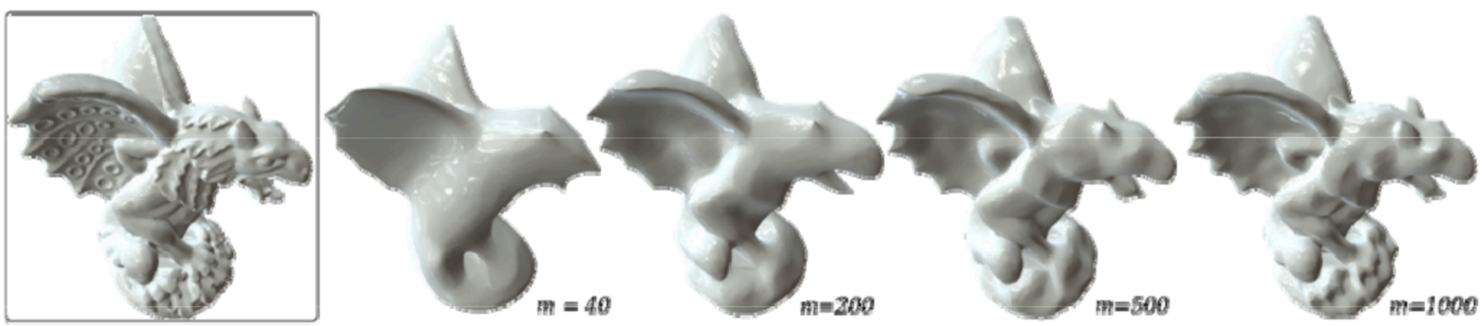
$\mathbf{v}_2$ $\mathbf{v}_{50}$

Demo

From Vallet et al., Eurographics 2008

Smoothing using the Laplacian Eigen-decomposition

$$\mathbf{P}^{smooth} = \mathbf{V}\left(\mathbf{D}_m\right)\mathbf{V}^T\mathbf{P} \quad , \quad \mathbf{D}_m = \begin{pmatrix} k_1 & & & \\ & \cdots & & \\ & & k_m & \\ & & & 0 \end{pmatrix}$$



From Vallet et al., Eurographics 2008