Introduction to Differential Geometry

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Chapter 1

Calculus of Euclidean Maps

The analytic study of surfaces involves multi-variable calculus. We begin with a "brief review" of calculus in $\mathbb{R}^n$. Let

$$\begin{aligned}\mathbb{R}^n &= n-\text{dimensional Euclidean space}\\ &= \left\{(x^1, x^2, x^3, \ldots, x^n) : x^i \in I\!R\right\}.\end{aligned}$$

(Note the superscripts; this is standard and traditional notation in differential geometry stemming from tensor calculus.)

Standard inner product on $\mathbb{R}^n$: $x = (x^1, x^2, \ldots, x^n)$, $y = (y^1, y^2, \ldots, y^n)$ then,

$$\langle x, y\rangle = x \cdot y = x^1y^1 + x^2y^2 + \cdots + x^ny^n = \sum_{i=1}^{n} x^iy^i$$

Norm:

$$|x| = \sqrt{\langle x, x\rangle} = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}$$

$$= \sqrt{\sum_{i=1}^{n}(x^i)^2}$$

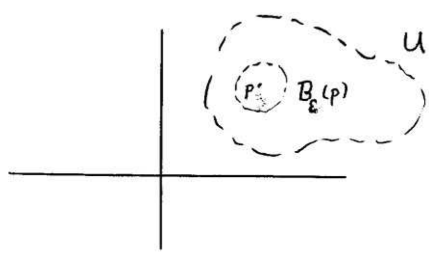
Distance Function on $\mathbb{R}^n$:

$$d(x,y) = |x-y| = \sqrt{(x^1-y^1)^2+(x^2-y^2)^2+\cdots+(x^n-y^n)^2}$$
$$= \sqrt{\sum_{i=1}^{n}(x^i-y^i)^2}$$

Open sets in $\mathbb{R}^n$:

$$B_r(p) = \text{open ball of radius } r \text{ centered at } p$$
$$= \{x \in \mathbb{R}^n : d(x,p) < r\}.$$

Definition. $U \subset \mathbb{R}^n$ is *open* provided for each $p \in U$ there exists $\epsilon > 0$ such that $B_\varepsilon(p) \subset U$.



Euclidean Mappings: $F : \mathbb{R}^n \to \mathbb{R}^m$

These are the types of maps that will arise most frequently in our study, for example.

1) $F : \mathbb{R} \to \mathbb{R}^3$: parameterized curve in space, $F(t) = (x(t), y(t), z(t))$, 1-parameter map.

2) $F : \mathbb{R}^2 \to \mathbb{R}^3$: parameterized surface in space, $F(u,v) = (x(u,v), y(u,v), z(u,v))$, 2-parameter map.

3) $F : \mathbb{R}^2 \to \mathbb{R}^2$: change of coordinates, e.g. polar coordinates,

$$F : \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta, \end{array}$$

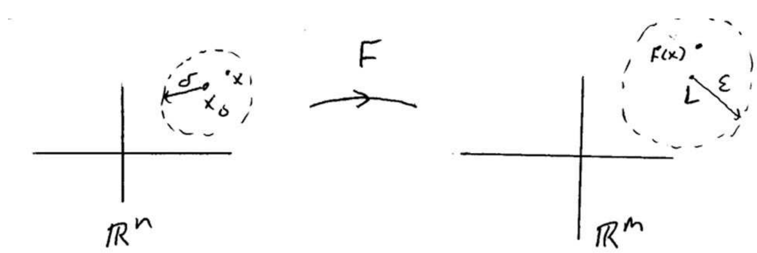
$F(r,\theta) = (r\cos\theta, r\sin\theta)$.

Limits and Continuity:

Definition. Consider the map $F : \mathbb{R}^n \to \mathbb{R}^m$. Assume F is defined in a "deleted" neighborhood of $x_0 \in \mathbb{R}^n$. Then,

$$\lim_{x \to x_0} F(x) = L$$

means that for every $\epsilon > 0$ there exists $\delta > 0$ such that, $|F(x) - L| < \epsilon$ whenever $|x - x_0| < \delta \quad (x \neq x_0)$.



Definition. $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. F is continuous at $x_0 \in U$ provided,

$$\lim_{x \to x_0} F(x) = F(x_0).$$

F is continuous on U if it is continuous at each point of U.

The following fact gives a useful chararcterization of continuity.

Proposition 1.1. *$F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on U iff for all open sets $V \subset \mathbb{R}^m$, $F^{-1}(V)$ is open in $\mathbb{R}^n$.*

Component Functions:

Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ it is often useful to express F in terms of its component functions:

$$\begin{aligned} F(x^1, \ldots, x^n) &= (y^1, \ldots, y^m) \\ &= (f^1(x^1, \ldots, x^n), ..., f^m(x^1, \ldots, x^n)) \\ &= (f^1(x), \ldots, f^m(x)), \qquad x = (x^1, \ldots, x^n). \end{aligned}$$

Then the component functions of F are: $f^i : U \subset \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$, which we sometimes display as,

$$F : \begin{array}{rcl} y^1 & = & f^1(x^1, \ldots, x^n) \\ y^2 & = & f^2(x^1, \ldots, x^n) \\ & \vdots & \\ y^m & = & f^m(x^1, \ldots, x^n) \end{array}$$

or,

$$F : y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, m.$$

Example. $F(x^1, x^2) = (2x^1x^2, x^2 - x^1)$

$$F : \begin{array}{rcl} y^1 & = & 2x^1x^2 \\ y^2 & = & x^2 - x^1 \end{array}$$

Example. $F : \mathbb{R}^2 \to \mathbb{R}^3, \quad F(u, v) = (\underbrace{uv^2}_{x}, \underbrace{u\cos v}_{y}, \underbrace{e^{u/v}}_{z})$

$$F : \begin{array}{rcl} x & = & uv^2 \\ y & = & u\cos v \\ z & = & e^{u/v} \end{array}$$

Proposition 1.2. *$F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on U if and only if its component functions $f^i : U \subset \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, is continuous on U.*

Differentiation of Mappings

Definition. Given $f : U \subset \mathbb{R}^n \to \mathbb{R}$. f is C^k on U provided f and its partial derivatives of order up to, and including, k are continuous on U. f is C^∞ on U (or *smooth* on U) provided f and its partial derivatives of all orders exist and are continuous on U.

Example. $f : U \subset \mathbb{R}^2 \to \mathbb{R}$. $\quad f(x, y) \quad f$ is C^2 means that $f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}, \dfrac{\partial^2 f}{\partial y^2}$ exist and are continuous on U.

Example. $f : \mathbb{R}^2 \to \mathbb{R}$. $f(x, y) = x^2 + 3xy - y^2$. f is C^∞ on $\mathbb{R}^2$.

Example. $f(x, y) = \ln(1 - x^2 - y^2)$. f is C^∞ on $U = \{(x, y) : x^2 + y^2 < 1\}$.

EXERCISE 1.1. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is C^1 but not C^2.

Definition. Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. F is C^k on U iff its component functions $f^1, \ldots, f^m$ are C^k on U. F is C^∞ (smooth) on U iff $f^1, \ldots, f^m$ are C^∞ (smooth) on U.

Example. $F : \mathbb{R}^2 \to \mathbb{R}^3$, $F(x, y) = (x \cos y, x \sin y, e^{xy})$. $f^1(x, y) = x \cos y, f^2(x, y) = x \sin y, f^3(x, y) = e^{xy}$ are smooth. Therefore F is smooth.

Remark. We will usually assume the mappings we deal with are smooth - even though some results might be true with weaker differentiability assumptions.

<u>Chain Rule for real valued functions of several variables:</u>

Given a smooth function of n variables, $w = f(x^1, \ldots, x^n)$ where $x^i = x^i(t, \ldots)$, $i = 1, \cdots, n$, depend smoothly on t. Then the composition $w = f(x^1(t, \ldots), \ldots, x^n(t, \ldots))$ depends smoothly on t and,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x^1}\frac{\partial x^1}{\partial t} + \frac{\partial w}{\partial x^2}\frac{\partial x^2}{\partial t} + \cdots + \frac{\partial w}{\partial x^n}\frac{\partial x^n}{\partial t}$$

or, using summation notation,

$$\frac{\partial w}{\partial t} = \sum_{i=1}^{n} \frac{\partial w}{\partial x^i} \cdot \frac{\partial x^i}{\partial t}\ .$$

<u>Jacobians:</u>

Definition. *Given $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ smooth with component functions,*

$$F : y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, m.$$

($\Leftrightarrow f^i : U \subset \mathbb{R}^n \to \mathbb{R}$ smooth), the Jacobian Matrix of F is the $m \times n$ matrix,

$$DF = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^n} \\ & \vdots & & \vdots \\ \frac{\partial y^m}{\partial x^1} & \frac{\partial y^m}{\partial x^2} & \cdots & \frac{\partial y^m}{\partial x^n} \end{bmatrix},$$

or, in short hand,

$$DF = \left[\frac{\partial y^i}{\partial x^j}\right]_{\substack{1\le i\le m \\ 1\le j\le n}}.$$

At $p \in \mathbb{R}^n$,

$$DF(p) = \left[\frac{\partial y^i}{\partial x^j}(p)\right].$$

Alternative notation: $J(F) = DF$.

Example. $F : \mathbb{R}^2 \to \mathbb{R}^3$, $F(x,y) = (\underbrace{x^2+y^2}_{y^1}, \underbrace{2xy}_{y^2}, \underbrace{x\cos y}_{y^3})$. *$DF$ is 3×2:*

$$DF = \begin{bmatrix} 2x & 2y \\ 2y & 2x \\ \cos y & -x\sin y \end{bmatrix}.$$

Remark. For ordinary real function $f : \mathbb{R} \to \mathbb{R}$, $y = f(x)$, DF is just the 1×1 matrix $\left[\frac{dy}{dx}\right]$, and so DF is essentially just the derivative of $y = f(x)$. For mappings, the Jacobian plays the role of first derivative.

Jacobian Determinant: Consider special case $m = n$. $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$F : y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n.$$

Then DF is a square $n \times n$ matrix. The Jacobian determinant is then defiend as,

$$\begin{aligned} \text{Jacobian determinant} &= \det DF \\ \frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)}. &= \det \left[\frac{\partial y^i}{\partial x^j}\right] \end{aligned}$$

Example. For the map, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x^2 - y^2, 2xy)$.

$$F : \begin{array}{rcl} u &=& x^2 - y^2 \\ v &=& 2xy \end{array}$$

DF is given by,

$$DF = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \det DF = 4(x^2 + y^2)$$

<u>Chain Rule for Mappings.</u>

The Calculus I chain rule may be written as follows: Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, then $(f \circ g)' = f' \cdot g'$.

Theorem 1.3 (Chain rule)**.** *Given smooth maps $F : V \subset \mathbb{R}^m \to \mathbb{R}^\ell$, $G : U \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $G(U) \subset V$. Then the composition $F \circ G : U \subset \mathbb{R}^n \to \mathbb{R}^\ell$ is defined and smooth, and*

$$D(F \circ G)(x) = DF(G(x))DG(x)$$

or simply,

$$D(F \circ G)_{\ell \times n} = \underbrace{DF \cdot DG}_{\substack{\text{matrix multiplication} \\ \ell\times m \;\; m\times n}}$$

Proof. Apply the chain rule for real valued functions of several variables. First, recall, if $A = [a_{ij}]_{\ell \times m}$ and $B = [b_{ij}]_{m \times n}$ then the product matrix $C = AB = [c_{ik}]_{\ell \times n}$ has entries given by,

$$c_{ik} = \sum_j a_{ij} b_{jk}$$

(i^{th} row of A dotted into k^{th} column of B)

Now, express F, G and $F \circ G$ in terms of component functions:

$$\begin{aligned} F : z^i &= f^i(y^1, \ldots, y^m), \;\; 1 \leq i \leq \ell \\ G : y^j &= g^j(x^1, \ldots, x^n), \;\; 1 \leq j \leq m \\ F \circ G : z^i &= f^i(g^1(x^1, \ldots, x^n), \ldots, g^m(x^1, \ldots, x^n)), 1 \leq i \leq \ell. \end{aligned}$$

For each $1 \leq k \leq n : z^i$ depends on the y^j's and the y^j's depend on x^k. Therefore z^i depends on x^k and by the CR for real valued functions of several variables,

$$\frac{\partial z^i}{\partial x^k} = \sum_{j=1}^m \frac{\partial z^i}{\partial y^j} \, \frac{\partial y^j}{\partial x^k}$$

The term being summed is the i, kth entry of the matrix product,

$$\left[\frac{\partial z^i}{\partial y^j}\right] \cdot \left[\frac{\partial y^j}{\partial x^k}\right],$$

and hence,

$$\left[\frac{\partial z^i}{\partial x^k}\right] = \left[\frac{\partial z^i}{\partial y^j}\right] \cdot \left[\frac{\partial y^j}{\partial x^k}\right],$$

or,

$$D(F \circ G) = DF \cdot DG.$$

The Inverse Function Theorem

Analytically the Jacobian of $F : \mathbb{R}^n \to \mathbb{R}^m$ plays a role analogous to f' for functions $f : \mathbb{R} \to \mathbb{R}$. For example just as the derivative can be used to approximate f,

$$f(x + \triangle x) \approx f(x) + f'(x)\triangle x,$$

the Jacobian can be used to approximate $F : \mathbb{R}^n \to \mathbb{R}^m$,

$$F(p + \triangle p) \approx F(p) + \underbrace{DF(p)}_{m\times n}\underbrace{\triangle p}_{n\times 1}$$

(provided F is C^1 - this all can be made very precise). In the above expression we are treating points in $\mathbb{R}^n$ and $\mathbb{R}^m$ as column vectors.

Recall, given a smooth function $f : \mathbb{R} \to \mathbb{R}$, if $f'(x_0) \neq 0$ then on a small interval I about x_0, f is either increasing ($f'(x_0) > 0$) or decreasing ($f'(x_0) < 0$). In either case f has an inverse f^{-1} on I and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = [f'(f^{-1}(y))]^{-1}$$

or, more simply,

$$(f^{-1})' = \frac{1}{f'} = (f')^{-1}$$

or, in differential notation, if $y = f(x)$ then $x = f^{-1}(y)$ and,

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \left(\frac{dy}{dx}\right)^{-1}.$$

Theorem 1.4 (Inverse function Theorem)**.** . *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Suppose for some $p \in U$, $DF(p)$ is nonsingular ($\Leftrightarrow$ $\det DF(p) \neq 0$). Then there is a nbd V of p such that*

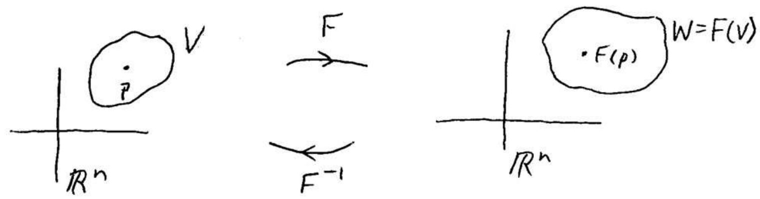
1. *$W = F(V)$ is open.*
2. *$F : V \to W$ is one-to-one and onto, and $F^{-1} : W \to V$ is smooth.*

3. For each $q \in W$*,*

$$D(F^{-1})(q) = [DF(F^{-1}(q))]^{-1},$$

or simply,

$$D(F^{-1}) = (DF)^{-1}.$$



EXERCISE 1.2. Assuming (1) and (2) hold, show that (3) necessarily holds. Hint: Differentiate both sides of the equation: $F \circ F^{-1} = id$ (where $id : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map, $id(x) = x$ for all $x \in \mathbb{R}^n$)

Remark. Let V, W be open sets in $\mathbb{R}^n$. A map $F : V \to W$ is called a *diffeomorphism* provided it is 1-1 and onto, and both F and F^{-1} are smooth. Conditions (1) and (2) in the IFT say that $F : V \to W$ is a diffeomorphism.

Remark. Let's specialize the statement of the IFT to the case $n = 2$. Hence, consider $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (u(x, y), v(x, y))$, i.e. F has component functions

$$F : \begin{array}{rcl} u &=& u(x,y) \\ v &=& v(x,y) \end{array} \qquad (x,y) \in U \tag{$*$}$$

F is smooth $\Leftrightarrow u = u(x, y),\ v = v(x, y)$ are smooth, and so,

$$DF = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

Let $p = (x_0, y_0) \in U$ be such that $\det DF(x_0, y_0) \neq 0$, and hence,

$$\det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \neq 0 \;\text{ at }\; (x_0, y_0).$$

Then, according to the IFT, there exists a neighborhood V of (x_0, y_0) such that $W = F(V)$ is an open set in the u-v plane, and $F^{-1} : W \to V$ is defined and smooth. We have $F^{-1}(u, v) = (x(u, v), y(u, v))$, i.e. F^{-1} has component functions,

$$F^{-1} : \begin{array}{c} x = x(u, v) \\ y = y(u, v) \end{array} \quad (u, v) \in W$$

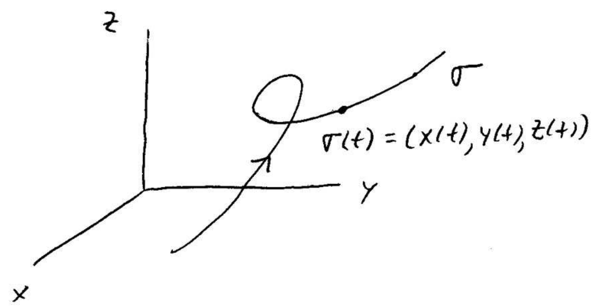
i.e. the equations $(*)$ can be smoothly inverted to obtain x and y in terms of u and v. Moreover, when evaluated at the appropriate points,

$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}^{-1}$$

Chapter 2

Parameterized Curves in $\mathbb{R}^3$

Definition. *A smooth curve in* $\mathbb{R}^3$ *is a smooth map* $\sigma : (a, b) \to \mathbb{R}^3$.

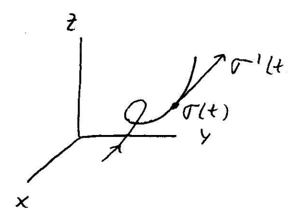


For each $t \in (a, b)$, $\sigma(t) \in \mathbb{R}^3$. As t increases from a to b, $\sigma(t)$ traces out a curve in $\mathbb{R}^3$. In terms of components,

$$\sigma(t) = (x(t), y(t), z(t)) \,, \tag{2.1}$$

or

$$\sigma : \begin{array}{l} x = x(t) \\ y = y(t) \\ z = z(t) \end{array} \quad a < t < b \,,$$



<u>velocity at time t:</u> $\dfrac{d\sigma}{dt}(t) = \sigma'(t) = (x'(t), y'(t), z'(t))$

<u>speed at time t:</u> $\left|\dfrac{d\sigma}{dt}(t)\right| = |\sigma'(t)|$

Example. $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (r\cos t, r\sin t, 0)$ - the standard parameterization of the unit circle,

$$\sigma : \begin{array}{l} x = r\cos t \\ y = r\sin t \\ z = 0 \end{array}$$

has velocity and speed,

$$\begin{aligned} \sigma'(t) &= (-r\sin t, r\cos t, 0) \\ |\sigma'(t)| &= r \qquad \text{(constant speed)} \end{aligned}$$

Example. $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (r\cos t, r\sin t, ht)$, $r, h > 0$ constants (the *helix*), has velocity and speed,

$$\begin{aligned} \sigma'(t) &= (-r\sin t, r\cos t, h) \\ |\sigma'(t)| &= \sqrt{r^2 + h^2} \quad \text{(constant)} \end{aligned}$$

Definition. A *regular* curve in $\mathbb{R}^3$ is a smooth curve $\sigma : (a, b) \to \mathbb{R}^3$ such that $\sigma'(t) \neq 0$ for all $t \in (a, b)$.

That is, a regular curve is a smooth curve with everywhere *nonzero velocity.* The examples above are regular.

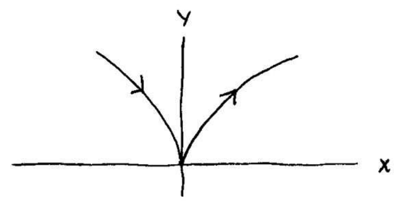
Example. $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (t^3, t^2, 0)$. σ is smooth, but not regular. σ has velocity,

$$\sigma'(t) = (3t^2, 2t, 0), \qquad \sigma'(0) = (0, 0, 0)$$

<u>Graph of σ:</u>

$$\sigma : \begin{array}{l} x = t^3 \\ y = t^2 \\ z = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} y = t^2 = (x^{1/3})^2 \\ y = x^{2/3} \end{array}$$

Hence, the graph is as follows,



There is a cusp, not because the curve isn't smooth, but because the velocity $= 0$ at the origin. A regular curve has a well-defined smoothly turning tangent, and hence its graph will appear smooth.

The Geometric Action of the Jacobian

Given a smooth map $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$, $p \in U$. Let X be any vector based at the point p. To X at p we associate a vector Y at $F(p)$ as follows.

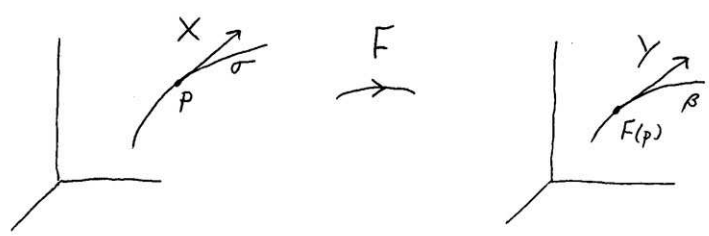
Let $\sigma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be any smooth curve such that,

$$\sigma(0) = p \quad \text{and} \quad \frac{d\sigma}{dt}(0) = X,$$

i.e. σ is a curve which passes through p at $t = 0$ with velocity X. (E.g. one can take $\sigma(t) = p + tX$.) Now, look at the image of σ under F, i.e. consider $\beta = F \circ \sigma$, $\beta : (-\epsilon, \epsilon) \to \mathbb{R}^3$, $\beta(t) = F \circ \sigma(t) = F(\sigma(t))$. We have, $\beta(0) = F(\sigma(0)) = F(p)$, i.e., β passes through $F(p)$ at $t = 0$. Finally, let

$$Y = \frac{d\beta}{dt}(0).$$

i.e. Y is the velocity vector of β at $t = 0$.



EXERCISE 2.1. Show that $Y = DF(p)X$.

Note: In the above, X and Y are represented as column vectors, and the right hand side of the equation involves matrix multiplication. Hint: Use the chain rule.

Thus, roughly speaking, the geometric effect of the Jacobian is to "send velocity vectors to velocity vectors". The same result holds for mappings $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ (i.e. it is not necessary to restrict to dimension three).

<u>Reparameterizations</u>

Given a regular curve $\sigma : (a, b) \to \mathbb{R}^3$. Traversing the same path at a different speed (and perhaps in the opposite direction) amounts to what is called a reparameterization.

Definition. Let $\sigma : (a, b) \to \mathbb{R}^3$ be a regular curve. Let $h : (c, d) \subset \mathbb{R} \to (a, b) \subset \mathbb{R}$ be a diffeomorphism (i.e. h is 1-1, onto such that h and h^{-1} are smooth). Then $\tilde{\sigma} = \sigma \circ h : (c, d) \to \mathbb{R}^3$, is a regular curve, called a reparameterization of σ.

I.e., start with a curve $\sigma = \sigma(t)$, make a change of parameter $t = h(u)$, then obtain the reparameterized curve $\tilde{\sigma}$ given by,

$$\tilde{\sigma}(u) = \sigma(h(u)) \,.$$

$t =$ original parameter, $u =$ new parameter.

Remarks:

1. σ and $\tilde{\sigma}$ describe the same path in space, just traversed at different speeds (and perhaps in opposite directions).

2. Compare velocities:

$$\begin{aligned} \tilde{\sigma} &= \sigma(h(u)) \quad \text{i.e.,} \\ \tilde{\sigma} &= \sigma(t), \ \text{where } t = h(u) \,. \end{aligned}$$

Hence, by the chain rule,

$$\frac{d\tilde{\sigma}}{du} = \frac{d\sigma}{dt} \cdot \frac{dt}{du} = \frac{d\sigma}{dt} \cdot h'$$

$h' > 0$: orientation preserving reparameterization.
$h' < 0$: orientation reversing reparameterization.

Example. $\sigma : (0, 2\pi) \to \mathbb{R}^3$, $\sigma(t) = (\cos t, \sin t, 0)$. Consider the reparameterization function, $h : (0, \pi) \to (0, 2\pi)$,

$$h : t = h(u) = 2u\,, \qquad u \in (0, \pi)\,.$$

Reparameterized curve:

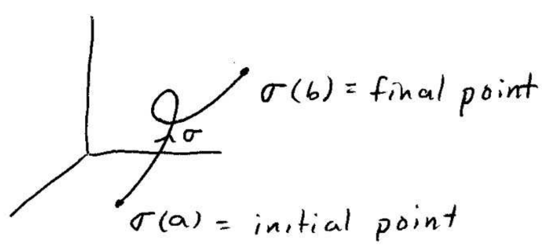
$$\begin{aligned}\tilde{\sigma}(u) &= \sigma(t) = \sigma(2u)\\ \tilde{\sigma}(u) &= (\cos 2u, \sin 2u, 0)\end{aligned}$$

$\tilde{\sigma}$ describes the same circle, but traversed twice as fast,

$$\text{speed of } \sigma = \left|\frac{d\sigma}{dt}\right| = 1\,, \qquad \text{speed of } \tilde{\sigma} = \left|\frac{d\tilde{\sigma}}{du}\right| = 2\,.$$

Remark. Regular curves always admit a very important reparameterization: they can always be parameterized in terms of arc length (see below).

Length Formula: Consider a smooth curve defined on a *closed* interval, $\sigma : [a, b] \to \mathbb{R}^3$.



σ is a smooth curve segment. Its *length* is defined by,

$$\text{length of } \sigma = \int_a^b |\sigma'(t)| dt.$$

I.e., to get the length, integrate speed *wrt* time.

Example. $\sigma(t) = (r \cos t,\ r \sin t, 0)\quad 0 \le t \le 2\pi$.

$$\text{Length of } \sigma = \int_0^{2\pi} |\sigma'(t)| dt = \int_0^{2\pi} r dt = 2\pi r.$$

Proposition 2.1. *The length formula is independent of parameterization, i.e., if* $\tilde{\sigma} : [c, d] \to \mathbb{R}^3$ *is a reparameterization of* $\sigma : [a, b] \to \mathbb{R}^3$ *then length of* $\tilde{\sigma}$ = *length of* σ.

EXERCISE 2.2. Prove this. (See e.g., [2, Prop. 2.1].)

<u>Arc Length Parameter:</u>

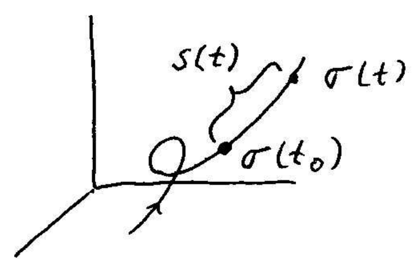
Along a regular curve $\sigma : (a, b) \to \mathbb{R}^3$ there is a distinguished parameter called *arc length* parameter. Fix $t_0 \in (a, b)$. Define the following function (*arc length function*).

$$s = s(t), \;\; t \in (a, b)\,, \qquad s(t) = \int_{t_0}^{t} |\sigma'(t)| dt\,.$$

Thus,

if $t > t_0$, $\;\; s(t) =$ length of σ from t_0 to t,

if $t < t_0$, $\;\; s(t) = -$length of σ from t_0 to t.



$s = s(t)$ is smooth and by the Fundamental Theorem of calculus,

$$s'(t) = |\sigma'(t)| > 0 \;\text{ for all } t \in (a, b)$$

Hence $s = s(t)$ is strictly increasing, and so has a smooth inverse - can solve smoothly for t in terms of s, $t = t(s)$ (reparameterization function).
Then,

$$\tilde{\sigma}(s) = \sigma(t(s))$$

is the arc length reparameterization of σ.

Proposition 2.2. *A regular curve admits a reparameterization in terms of arc length.*

Example. Reparameterize the circle $\sigma(t) = (r\cos t, r\sin t, 0)$, $-\infty < t < \infty$, in terms of arc length parameter.

Obtain the arc length function $s = s(t)$,

$$\begin{aligned} s &= \int_0^t |\sigma'(t)|dt = \int_0^t r dt \\ s &= rt \quad \Rightarrow \quad t = \frac{s}{r} \quad \text{(reparam. function)} \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{\sigma}(s) &= \sigma(t(s)) = \sigma\left(\frac{s}{r}\right) \\ \tilde{\sigma}(s) &= (r\cos\left(\frac{s}{r}\right),\ r\sin\left(\frac{s}{r}\right), 0). \end{aligned}$$

Remarks:

1. Often one relaxes the notation and writes $\sigma(s)$ for $\tilde{\sigma}(s)$ (i.e. one drops the tilde).

2. Let $\sigma = \sigma(t)$, $t \in (a, b)$ be a *unit speed curve*, $|\sigma'(t)| = 1$ for all $t \in (a, b)$. Then,

$$\begin{aligned} s &= \int_{t_0}^t |\sigma'(t)|dt = \int_{t_0}^t 1 dt \\ s &= t - t_0\,. \end{aligned}$$

I.e. up to a trivial translation of parameter, $s = t$. Hence unit speed curves are already parameterized *wrt* arc length (as measured from some point). Conversely, if $\sigma = \sigma(s)$ is a regular curve parameterized *wrt* arc length s then σ is unit speed, i.e. $|\sigma'(s)| = 1$ for all s (why?). Hence the phrases "unit speed curve" and "curve parameterized *wrt* arc length" are used interchangably.

EXERCISE 2.3. Consider the helix,

$$\sigma(t) = (r\cos t, r\sin t, ht).$$

Show that, when parameterized wrt arc length, we obtain,

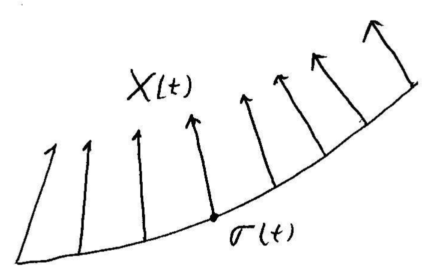
$$\sigma(s) = (r\cos\omega s, r\sin\omega s, h\omega s)\,.$$

where $\omega = \dfrac{1}{\sqrt{r^2+h^2}}$.

Vector fields along a curve.

We will frequently use the notion of a vector field along a curve σ.

Definition. Given a smooth curve $\sigma : (a,b) \to \mathbb{R}^3$ a *vector field along* σ is a vector-valued map $X : (a,b) \to \mathbb{R}^3$ which assigns to each $t \in (a,b)$ a vector $X(t)$ at the point $\sigma(t)$.

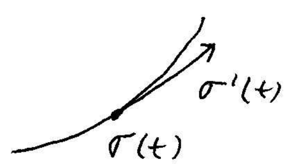


Some examples:

1. The velocity vector field along $\sigma : (a,b) \to \mathbb{R}^3$.

$$\sigma' : (a,b) \to \mathbb{R}^3, \quad t \to \sigma'(t)\,.$$

If $\sigma(t) = (x(t), y(t), z(t))$, $\sigma'(t) = (x'(t), y'(t), z'(t))$.



2. The unit tangent vector field along σ,

$$T(t) = \frac{\sigma'(t)}{|\sigma'(t)|}\,,$$

$|T(t)| = 1$ for all t. (Note σ must be regular for T to be defined).

Example. Find the unit tangent vector field along $\sigma(t) = (r\cos t, r\sin t, ht)$.

$$\begin{aligned}
\sigma'(t) &= (-r\sin t, r\cos t, h) \\
|\sigma'(t)| &= \sqrt{r^2+h^2} \\
T(t) &= \frac{1}{\sqrt{r^2+h^2}}(-r\sin t, r\cos t, h)
\end{aligned}$$

Note. If $s \to \sigma(s)$ is parameterized *wrt* arc length then $|\sigma'(s)| = 1$ (unit speed) and so,

$$T(s) = \sigma'(s)\,. \tag{2.2}$$

Differentiation. Analytically, vector fields along a curve are just maps,

$$X : (a,b) \subset \mathbb{R} \to \mathbb{R}^3.$$

Can differentiate by expressing $X = X(t)$ in terms of components,

$$\begin{aligned}
X(t) &= (X^1(t), X^2(t), X^3(t))\,, \\
\frac{dX}{dt} &= \left(\frac{dX^1}{dt}, \frac{dX^2}{dt}, \frac{dX^3}{dt}\right)\,.
\end{aligned}$$

Example. Consider the unit tangent field to the helix,

$$\begin{aligned}
T(t) &= \frac{1}{\sqrt{r^2+h^2}}(-r\sin t, r\cos t, h) \\
T'(t) &= \frac{1}{\sqrt{r^2+h^2}}(-r\cos t, -r\sin t, 0).
\end{aligned}$$

EXERCISE 2.4. Let $X = X(t)$ and $Y = Y(t)$ be two smooth vector fields along $\sigma : (a,b) \to \mathbb{R}^3$. Prove the following product rules,

$$(1) \quad \frac{d}{dt}\langle X, Y\rangle = \langle \frac{dX}{dt}, Y\rangle + \langle X, \frac{dY}{dt}\rangle$$

$$(2) \quad \frac{d}{dt} X \times Y = \frac{dX}{dt} \times Y + X \times \frac{dY}{dt}$$

Hint: Express in terms of components.

Curvature

Curvature of a curve is a measure of how much a curve bends at a given point:

This is quantified by measuring the rate at which the unit tangent turns *wrt distance* along the curve. Given a regular curve, $t \to \sigma(t)$, reparameterize in terms of arc length, $s \to \sigma(s)$, and consider the unit tangent vector field,

$$T = T(s) \qquad (T(s) = \sigma'(s)).$$

Now differentiate $T = T(s)$ *wrt* arc length,

$$\frac{dT}{ds} = \text{curvature vector}$$

.

The *direction* of $\frac{dT}{ds}$ tells us which way the curve is bending. Its magnitude tells us *how much* the curve is bending,

$$\left|\frac{dT}{ds}\right| = \text{curvature}$$

Definition. Let $s \to \sigma(s)$ be a unit speed curve. The curvature $\kappa = \kappa(s)$ of σ is defined as follows,

$$\kappa(s) = |T'(s)| \;\; (= |\sigma''(s)|)\,,$$

where $' = \frac{d}{ds}$.

Example. Compute the curvature of a circle of radius r.

Standard parameterization: $\quad \sigma(t) = (r\cos t, r\sin t, 0)$.

Arc length parameterization: $\sigma(s) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right), 0\right)$.

Then we compute,

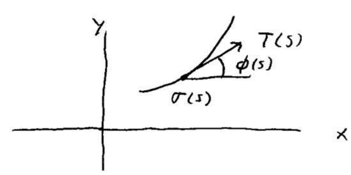
$$\begin{aligned}
T(s) &= \sigma'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0\right) \\
T'(s) &= \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right), 0\right) \\
&= -\frac{1}{r}\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\right) \\
\kappa(s) &= |T'(s)| = \frac{1}{r}
\end{aligned}$$

(Does this answer agree with intuition?)

EXERCISE 2.5. Let $s \to \sigma(s)$ be a unit speed *plane* curve,

$$\sigma(s) = (x(s), y(s), 0)\,.$$

For each s let $\phi(s)$ = angle between positive x-axis and $T(s)$



Show: $\kappa(s) = |\phi'(s)| = \left|\dfrac{d\phi}{ds}\right|$. Hint: Observe, $T(s) = \cos\phi(s)\mathbf{i} + \sin\phi(s)\mathbf{j}$ (why?).

Conceptually, the definition of curvature is the right one. But for computational purposes it's not so good. For one thing, it would be useful to have a formula for computing curvature which does not require that the curve be parameterized with respect to arc length. Using the chain rule, such a formula is easy to obtain.

Given a regular curve $t \to \sigma(t)$, it can be reparameterized *wrt* arc length $s \to \sigma(s)$. Let $T = T(s)$ be the unit tangent field to σ; we have,

$$T = T(s), \quad s = s(t)\,.$$

So by the chain rule,

$$\begin{aligned} \frac{dT}{dt} &= \frac{dT}{ds}\cdot\frac{ds}{dt} \\ &= \frac{dT}{ds}\left|\frac{d\sigma}{dt}\right| \\ \left|\frac{dT}{dt}\right| &= \left|\frac{d\sigma}{dt}\right|\underbrace{\left|\frac{dT}{ds}\right|}_{\kappa} \end{aligned}$$

and hence,

$$\kappa = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{d\sigma}{dt}\right|},$$

i.e.,

$$\kappa(t) = \frac{|T'(t)|}{|\sigma'(t)|}, \quad ' = \frac{d}{dt}. \tag{2.3}$$

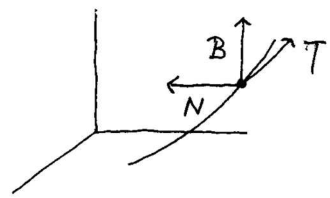
EXERCISE 2.6. Use the above formula to compute the curvature of the helix $\sigma(t) = (r\cos t, r\sin t, ht)$.

Frenet Equations

Let $s \to \sigma(s)$, $s \in (a, b)$ be a regular unit speed curve such that $\kappa(s) \neq 0$ for all $s \in (a, b)$. (We will refer to such a curve as *strongly* regular). Along σ we are going to introduce the vector fields,

$T = T(s)$ - unit tangent vector field
$N = N(s)$ - principal normal vector field
$B = B(s)$ - binormal vector field

$\{T, N, B\}$ is called a Frenet frame. At each point of σ $\{T, N, B\}$ forms an orthonormal basis, i.e. T, N, B are mutually perpendicular unit vectors (see figure next page).



To begin the construction of the Frenet frame, we have the unit tangent vector field,

$$T(s) = \sigma'(s), \qquad ' = \frac{d}{ds}$$

Consider the derivative $T' = T'(s)$.

Claim. $T' \perp T$ along σ.

Proof. It suffices to show $\langle T', T \rangle = 0$ for all $s \in (a, b)$. Along σ,

$$\langle T, T \rangle = |T|^2 = 1.$$

Differentiating both sides,

$$\begin{aligned} \frac{d}{ds}\langle T, T \rangle = \frac{d}{ds}1 &= 0 \\ \langle \frac{dT}{ds}, T \rangle + \langle T, \frac{dT}{ds} \rangle &= 0 \\ 2\langle \frac{dT}{ds}, T \rangle &= 0 \\ \langle T', T \rangle &= 0. \end{aligned}$$

Definition. Let $s \to \sigma(s)$ be a strongly regular unit speed curve. The *principal normal* vector field along σ is defined by

$$N(s) = \frac{T'(s)}{|T'(s)|} = \frac{T'(s)}{\kappa(s)} \qquad (\kappa(s) \neq 0) \tag{2.4}$$

The *binormal* vector field along σ is defined by

$$B(s) = T(s) \times N(s). \tag{2.5}$$

Note, the definition of $N = N(s)$ implies the equation

$$T' = \kappa N\,. \tag{2.6}$$

Claim. For each s, $\{T(s), N(s), B(s)\}$ is an orthonormal basis for vectors in space based at $\sigma(s)$.

Mutually perpendicular:

$$\langle T, N\rangle = \langle T, \frac{T'}{\kappa}\rangle = \frac{1}{\kappa}\langle T, T'\rangle = 0.$$

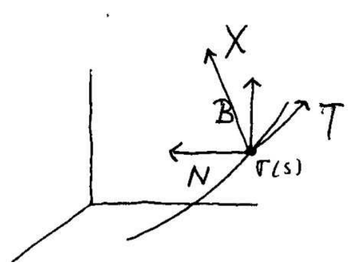
$$B = T \times N \Rightarrow \langle B, T\rangle = \langle B, N\rangle = 0.$$

Unit length: $|T| = 1$, and

$$\begin{aligned}
|N| &= \left|\frac{T'}{|T'|}\right| = \frac{|T'|}{|T'|} = 1,\\
|B|^2 &= |T \times N|^2\\
&= |T|^2|N|^2 - \langle T, N\rangle^2 = 1.
\end{aligned}$$

Remark on orthonormal bases.

Let X be a vector at the point $\sigma(s)$:



Then X can be expressed as a linear combination

$$X = aT + bN + cB$$

The constants a, b, c are determined as follows,

$$\begin{aligned}
\langle X, T\rangle &= \langle aT + bN + cB, T\rangle\\
&= a\langle T, T\rangle + b\langle N, T\rangle + c\langle B, T\rangle\\
&= a\,.
\end{aligned}$$

Hence, $a = \langle X, T\rangle$, and similarly, $b = \langle X, N\rangle$, $c = \langle X, B\rangle$. Hence X can be expressed as,

$$X = \langle X, T\rangle T + \langle X, N\rangle N + \langle X, B\rangle B. \tag{2.7}$$

Torsion: Torsion is a measure of "twisting". Curvature is associated with T'; torsion is associated with B':

$$\begin{aligned} B &= T \times N \\ B' &= T' \times N + T \times N' \\ &= \kappa N \times N + T \times N' \end{aligned}$$

Therefore $B' = T \times N'$ which implies $B' \perp T$, i.e.

$$\langle B', T\rangle = 0$$

Also, since $B = B(s)$ is a unit vector along σ, $\langle B, B\rangle = 1$, differentiation and the metric product rule imply

$$\langle B', B\rangle = 0$$

It follows that B' is a multiple of N,

$$\begin{aligned} B' &= \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B \\ &= \langle B', N\rangle N. \end{aligned}$$

Hence, we may write,

$$B' = -\tau N\,, \tag{2.8}$$

where,

$$\tau = \textit{torsion} \; = -\langle B', N\rangle\,. \tag{2.9}$$

Remarks.

1. τ is a *function* of s, $\tau = \tau(s)$.
2. τ is *signed* i.e. can be positive or negative.
3. $|\tau(s)| = |B'(s)|$, i.e., $\tau = \pm|B'|$, and hence τ measuures how B wiggles.

Given a strongly regular unit speed curve σ, the collection of quantities T, N, B, κ, τ is sometimes referred to as the *Frenet apparatus.*

Example. Compute T, N, B, κ, τ for the unit speed circle.

$$\begin{aligned}
\sigma(s) &= \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right), 0\right) \\
T &= \sigma' = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0\right) \\
T' &= -\frac{1}{r}\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\right)
\end{aligned}$$

$$\begin{aligned}
\kappa &= |T'| = \frac{1}{r} \\
N &= \frac{T'}{k} = -\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\right) \\
B &= T \times N \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -s & c & 0 \\ -c & -s & 0 \end{vmatrix} \\
&= \mathbf{k} = (0, 0, 1),
\end{aligned}$$

(where $c = \cos\left(\frac{s}{r}\right)$ and $s = \sin\left(\frac{s}{r}\right)$). Finally, since $B' = 0$, $\tau = 0$, i.e. the torsion vanishes.

Conjecture. Let $s \to \sigma(s)$ be a strongly regular unit speed curve. Then, σ is a *plane curve* iff its torsion vanishes, $\tau \equiv 0$.

Example. Compute T, N, B, κ, τ for the unit speed helix.

By Exercise 2.3,

$$\sigma(s) = (r\cos\omega s, r\sin\omega s, h\omega s).$$

where $\omega = \dfrac{1}{\sqrt{r^2 + h^2}}$.

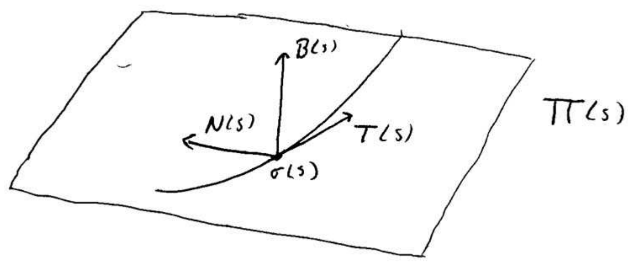
$$\begin{aligned}
T &= \sigma' = (-r\omega\sin\omega s, r\omega\cos\omega s, h\omega) \\
T' &= -\omega^2 r(\cos\omega s, \sin\omega s, 0) \\
\kappa &= |T'| = \omega^2 r = \frac{r}{r^2+h^2} = \text{const.} \\
N &= \frac{T'}{\kappa} = (-\cos\omega s, -\sin\omega s, 0)
\end{aligned}$$

$$\begin{aligned}
B &= T\times N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\omega\sin\omega s & r\omega\cos\omega s & h\omega \\ -\cos\omega s & -sin\omega s & 0 \end{vmatrix} \\
B &= (h\omega\sin\omega s, -h\omega\cos\omega s, r\omega) \\
B' &= (h\omega^2\cos\omega s, h\omega^2\sin\omega s, 0) \\
&= h\omega^2(\cos\omega s, \sin\omega s, 0) \\
B' &= -h\omega^2 N \\
B' &= -\tau N \Rightarrow \tau = hw^2 = \frac{h}{r^2+h^2}.
\end{aligned}$$

<u>The osculating plane:</u>

$\Pi(s) =$ *osculating plane* of σ at $\sigma'(s)$
$\quad =$ plane passing through $\sigma(s)$ spanned by $T(s)$ and $N(s)$
$\quad$ (or equivalently, perpendicular to $B(s)$)

See figure next page.



<u>*Comments.*</u>

(1) $s \to \Pi(s)$ is the family of osculating planes along σ. The Frenet equation $B' = -\tau N$ shows that the torsion τ measures how the osculating plane is twisting along σ.

(2) $\Pi(s_0)$ passes through $\sigma(s_0)$ and is spanned by $\sigma'(s_0)$ and $\sigma''(s_0)$. Hence, in a sense that can be made precise, $s \to \sigma(s)$ lies in $\Pi(s_0)$ "to second order in s". If $\tau(s_0) \neq 0$ then $\sigma'''(s_0)$ is *not* tangent to $\Pi(s_0)$. Hence the torsion τ gives a measure of the extent to which σ twists out of a given fixed osculating plane (see e.g. [3, p. 60f]).

Theorem 2.3. *(Frenet Formulas) Let $s \to \sigma(s)$ be a strongly regular unit speed curve. Then the Frenet frame, T, N, B satisfies,*

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

Proof. We have already established the first and third formulas (see equations (2.6) and (2.8)). To establish the second, observe $B = T \times N \Rightarrow N = B \times T$. Hence,

$$\begin{aligned} N' &= (B \times T)' = B' \times T + B \times T' \\ &= -\tau N \times T + \kappa B \times N \\ &= -\tau(-B) + \kappa(-T) \\ &= -\kappa T + \tau B. \end{aligned}$$

∎

We can express the Frenet formulas as a matrix equation,

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}}_{A} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

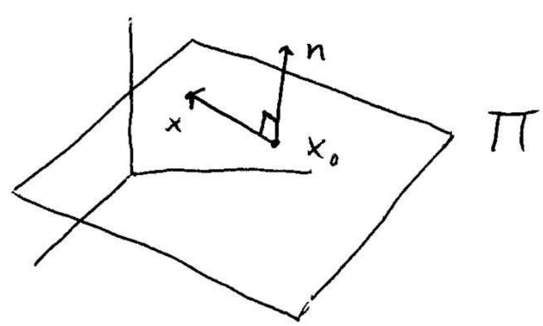
Note that A is skew symmetric: $A^t = -A$. $A = [a_{ij}]$, then $a_{ji} = -a_{ij}$.

The Frenet equations can be used to derive various properties of space curves.

Proposition 2.4. *Let $s \to \sigma(s)$, $s \in (a,b)$, be a strongly regular unit speed curve. Then, σ is a plane curve iff its torsion vanishes, $\tau \equiv 0$.*

Proof. Recall, the plane Π which passes through the point $x_0 \in \mathbb{R}^3$ and is perpendicular to the unit vector n consists of all points $x \in \mathbb{R}^3$ which satisfy the equation,

$$\langle n, x - x_0 \rangle = 0\,.$$



$\Rightarrow$: Assume $s \to \sigma(s)$ lies in the plane Π. Then, for all s,

$$\langle n, \sigma(s) - x_0 \rangle = 0$$

Since n is constant, differentiating twice gives,

$$\begin{aligned} \frac{d}{ds}\langle n, \sigma(s) - x_0 \rangle = \langle n, \sigma' \rangle = \langle n, T \rangle &= 0\,, \\ \frac{d}{ds}\langle n, T \rangle = \langle n, T' \rangle = \kappa \langle n, N \rangle &= 0\,, \end{aligned}$$

Since n is a unit vector perpendicular to T and N, $n = \pm B$, so $B = \pm n$. I.e., $B = B(s)$ is *constant* which implies $B' = 0$. Therefore $\tau \equiv 0$.

$\Leftarrow$: Now assume $\tau \equiv 0$. $B' = -\tau N \Rightarrow B' = 0$, i.e. $B(s)$ is constant,

$$B(s) = B = \text{ constant vector.}$$

We show $s \to \sigma(s)$ lies in the plane, $\langle B, x - \sigma(s_0)\rangle = 0$, passing through $\sigma(s_0)$, $s_0 \in (a, b)$, and perpendicular to B, i.e., will show,

$$\langle B, \sigma(s) - \sigma(s_0)\rangle = 0\,. \qquad (*)$$

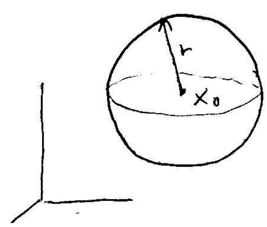
for all $s \in (a, b)$. Consider the function, $f(s) = \langle B, \sigma(s) - \sigma(s_0)\rangle$. Differentiating,

$$\begin{aligned} f'(s) &= \frac{d}{ds}\langle B, \sigma(s) - \sigma(s_0)\rangle \\ &= \langle B', \sigma(s) - \sigma(s_0)\rangle + \langle B, \sigma'(s)\rangle \\ &= 0 + \langle B, T\rangle = 0\,. \end{aligned}$$

Hence, $f(s) = c =$ const. Since $f(s_0) = \langle B, \sigma(s_0) - \sigma(s_0)\rangle = 0.$, $c = 0$ and thus $f(s) \equiv 0$. Therefore $(*)$ holds, i.e., $s \to \sigma(s)$ lies in the plane $\langle B, x - \sigma(s_0)\rangle = 0$. □

<u>Sphere Curves.</u> A sphere curve is a curve in $\mathbb{R}^3$ which lies on a sphere,

$$\begin{aligned} |x - x_0|^2 &= r^2\,, \qquad \text{(sphere of radius } r \text{ centered at } x_0) \\ \langle x - x_0, x - x_0\rangle &= r^2 \end{aligned}$$



Thus, $s \to \sigma(s)$ is a sphere curve iff there exists $x_0 \in \mathbb{R}^3$, $r > 0$ such that

$$\langle \sigma(s) - x_0, \sigma(s) - x_0\rangle = r^2\,, \quad \text{for all } s. \qquad (*)$$

If $s \to \sigma(s)$ lies on a sphere of radius r, it is reasonable to conjecture that σ has curvature $\kappa \geq \dfrac{1}{r}$ (why?). We prove this.

Proposition 2.5. *Let $s \to \sigma(s)$, $s \in (a,b)$, be a unit speed curve which lies on a sphere of radius r. Then its curvature function $\kappa = \kappa(s)$ satisfies, $\kappa \geq \dfrac{1}{r}$.*

Proof. Differentiating $(*)$ gives,

$$2\langle \sigma', \sigma - x_0 \rangle = 0$$

i.e.,

$$\langle T, \sigma - x_0 \rangle = 0.$$

Differentiating again gives:

$$\begin{aligned}
\langle T', \sigma - x_0 \rangle + \langle T, \sigma' \rangle &= 0 \\
\langle T', \sigma - x_0 \rangle + \langle T, T \rangle &= 0 \\
\langle T', \sigma - x_0 \rangle &= -1 \quad (\Rightarrow T' \neq 0) \\
\kappa \langle N, \sigma - x_0 \rangle &= -1
\end{aligned}$$

But,

$$\begin{aligned}
|\langle N, \sigma - x_0 \rangle| &= |N||\sigma - x_0||\cos\theta| \\
&= r|\cos\theta|,
\end{aligned}$$

and so,

$$\kappa = |\kappa| = \frac{1}{|\langle N, \sigma - x_0 \rangle|} = \frac{1}{r|\cos\theta|} \geq \frac{1}{r}$$

. ∎

EXERCISE 2.7. Prove that any unit speed sphere curve $s \to \sigma(s)$ having constant curvature is a circle (or part of a circle). (Suggestion: Show that the torsion vanishes (why is this sufficient?). To show this differentiate $(*)$ a few times.)

<u>Lancrets Theorem.</u>

Consider the unit speed circular helix $\sigma(s) = (r\cos\omega s, r\sin\omega s, h\omega s), \omega = 1/\sqrt{r^2+h^2}$. This curve makes a constant angle *wrt* the z-axis: We have, $T = \langle -r\omega\sin\omega s, r\cos\omega s, h\omega \rangle$, and hence,

$$\cos\theta = \frac{\langle T, \mathbf{k} \rangle}{|T||\mathbf{k}|} = h\omega = \text{const.}$$

Definition. A unit speed curve $s \to \sigma(s)$ is called a generalized helix if its unit tangent T makes a constant angle with a fixed unit direction vector $\mathbf{u}$ ($\Leftrightarrow \ \langle T, \mathbf{u}\rangle = \cos\theta = \text{const}$).

Theorem 2.6. *(Lancret) Let $s \to \sigma(s)$, $s \in (a,b)$ be a strongly regular unit speed curve such that $\tau(s) \neq 0$ for all $s \in (a,b)$. Then σ is a generalized helix iff κ/τ =constant.*

For a proof, see [2, p. 32].

Non-unit Speed Curves.

Given a strongly regular curve $t \to \sigma(t)$ (hence, $\kappa \neq 0$) it can be reparameterized in terms of arc length $s \to \tilde{\sigma}(s)$, $\tilde{\sigma}(s) = \sigma(t(s))$. The quantities T, N, B, κ, T can then be computed. But it is convenient to have formulas for these quantities which do not involve reparameterizing in terms of arc length.

Proposition 2.7. *Let $t \to \sigma(t)$ be a strongly regular curve in $\mathbb{R}^3$. Then*

(a) $T = \dfrac{\dot{\sigma}}{|\dot{\sigma}|}, \quad \cdot = \dfrac{d}{dt}$

(b) $B = \dfrac{\dot{\sigma} \times \ddot{\sigma}}{|\dot{\sigma} \times \ddot{\sigma}|}$

(c) $N = B \times T$

(d) $\kappa = \dfrac{|\dot{\sigma} \times \ddot{\sigma}|}{|\dot{\sigma}|^3}$

(e) $\tau = \dfrac{\langle \dot{\sigma} \times \ddot{\sigma}, \dddot{\sigma}\rangle}{|\dot{\sigma} \times \ddot{\sigma}|^2}$

Proof. We derive some of these. See e.g. [2, Section 2-6] for further details. Interpreting physically, we have t=time, $\dot{\sigma}$=velocity, $\ddot{\sigma}$=acceleration.

The unit tangent may be expressed as,

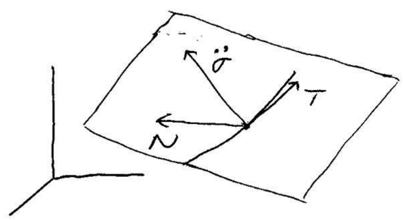
$$T = \frac{\dot{\sigma}}{|\dot{\sigma}|} = \frac{\dot{\sigma}}{v}$$

where $v = |\dot{\sigma}| = \text{speed}$.

Hence,

$$\begin{aligned}
\dot\sigma &= vT \\
\ddot\sigma &= \frac{d}{dt}vT = \frac{dv}{dt}T + v\frac{dT}{dt} \\
&= \frac{dv}{dt}T + v\frac{dT}{ds}\cdot\frac{ds}{dt} \\
&= \frac{dv}{dt}T + v(\kappa N)v \\
\ddot\sigma &= \dot v T + v^2\kappa N
\end{aligned}$$

Side Comment: This is the well-known expression for acceleration in terms of its tangential and normal components.



$$\begin{aligned}
\dot v \quad &= \text{tangential component of acceleration } (\dot v = \ddot s) \\
v^2\kappa \quad &= \text{normal component of acceleration} \\
&= \text{centripetal acceleration (for a circle, } v^2\kappa = \frac{v^2}{r}\text{)}.
\end{aligned}$$

$\dot\sigma, \ddot\sigma$ lie in the osculating plane; if $\tau \neq 0$, $\dddot\sigma$ does not.

Continuing the derivation,

$$\begin{aligned}
\dot\sigma \times \ddot\sigma &= vT \times (\dot v T + v^2\kappa N) \\
&= v\dot v T \times T + v^3\kappa T \times N \\
\dot\sigma \times \ddot\sigma &= v^3\kappa B \\
|\dot\sigma \times \ddot\sigma| &= v^3\kappa|B| = v^3\kappa
\end{aligned}$$

Hence,

$$\kappa = \frac{|\dot\sigma \times \ddot\sigma|}{v^3} = \frac{|\dot\sigma \times \ddot\sigma|}{|\dot\sigma|^3}$$

Also,

$$B = const \cdot \dot{\sigma} \times \ddot{\sigma} = \frac{\dot{\sigma} \times \ddot{\sigma}}{|\dot{\sigma} \times \ddot{\sigma}|}\,.$$

EXERCISE 2.8. Derive the expression for τ. Hint: Compute $\dddot{\sigma}$ and use Frenet formulas.

EXERCISE 2.9. Suppose σ is a regular curve in the x-y plane, $\sigma(t) = (x(t), y(t), 0)$, i.e.,

$$\sigma : \begin{array}{l} x = x(t) \\ y = y(t) \end{array}$$

(a) Show that the curvature of σ is given by,

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

(b) Use this formula to compute the curvature $\kappa = \kappa(t)$ of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

The Fundamental Theorem of Space Curves

This theorem says basically that any strongly regular unit speed curve is completely determined by its curvature and torsion (up to a Euclidean motion).

Theorem 2.8. *Let $\overline{\kappa} = \overline{\kappa}(s)$ and $\overline{\tau} = \overline{\tau}(s)$ be smooth functions on an interval (a, b) such that $\overline{\kappa}(s) > 0$ for all $s \in (a, b)$. Then there exists a strongly regular unit speed curve $s \to \sigma(s)$, $s \in (a, b)$ whose curvature and torsion functions are $\overline{\kappa}$ and $\overline{\tau}$, respectively. Moreover, σ is essentially unique, i.e. any other such curve $\tilde{\sigma}$ can be obtained from σ by a Euclidean motion (translation and/or rotation).*

Remarks.

1. The FTSC shows that curvature and torsion are the *essential* quantities for describing space curves.

2. The FTSC also illustrates a very important issue in differential geometry. The problem of establishing the existence of some geometric object

having certain geometric properties often reduces to a problem concerning the existence of a solution to some differential equation, or system of differential equations.

Proof. Fix $s_0 \in (a, b)$, and in space fix $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and a positively oriented orthonormal frame of vectors at P_0, $\{T_0, N_0, B_0\}$.

We show that there exists a *unique* unit speed curve $\sigma : (a, b) \to \mathbb{R}^3$ having curvature $\overline{\kappa}$ and torsion $\overline{\tau}$ such that $\sigma(s_0) = P_0$ and σ has Frenet frame $\{T_0, N_0, B_0\}$ at $\sigma(s_0)$.

The proof is based on the Frenet formulas:

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

or, in matrix form,

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

The idea is to mimmick these equations using the *given functions* $\overline{\kappa}$, $\overline{\tau}$. Consider the following *system* of O.D.E.'s in the (as yet unknown) vector-valued functions $e_1 = e_1(s), e_2 = e_2(s), e_3 = e_3(s)$,

$$\left.\begin{aligned} \frac{de_1}{ds} &= \overline{\kappa} e_2 \\ \frac{de_2}{ds} &= -\overline{\kappa} e_1 + \overline{\tau} e_3 \\ \frac{de_3}{ds} &= -\overline{\tau} e_2 \end{aligned}\right\} (*)$$

We express this system of ODE's in a notation convenient for the proof:

$$\frac{d}{ds}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \overline{\kappa} & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} \\ 0 & -\overline{\tau} & 0 \end{bmatrix}}_{\Omega}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

Set,

$$\Omega = \begin{bmatrix} 0 & \overline{\kappa} & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} \\ 0 & -\overline{\tau} & 0 \end{bmatrix} = \left[\Omega_i{}^j\right],$$

i.e. $\Omega_1{}^1 = 0$, $\Omega_1{}^2 = \overline{\kappa}$, $\Omega_1{}^3 = 0$, etc. Note that Ω is skew symmetric, $\Omega^t = -\Omega \iff \Omega_j{}^i = -\Omega_i{}^j$, $1 \le i,j \le 3$. Thus we may write,

$$\frac{d}{ds}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \left[\Omega_i{}^j\right]\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

or,

$$\frac{d}{ds}e_i = \sum_{j=1}^{3}\Omega_i{}^j e_j\,, \quad 1 \le i \le 3\,. \tag{2.10}$$

Consider the sytem of ODE's (2.10) subject to the initial conditions,

$$\text{IC}: \quad \begin{aligned} e_1(s_0) &= T_0 \\ e_2(s_0) &= N_0 \\ e_3(s_0) &= B_0 \end{aligned}$$

Basic existence and unique result for systems of linear ODE's guarantees that this system has a unique solution:

$$s \to e_1(s),\ s \to e_2(s),\ s \to e_3(s),\ s \in (a,b)\,.$$

We show that $e_1 = T$, $e_2 = N$, $e_3 = B$, $\overline{\kappa} = \kappa$ and $\overline{\tau} = \tau$ for some unit speed curve $s \to \sigma(s)$.

Claim. $\{e_1(s), e_2(s), e_3(s)\}$ is an orthonormal frame for all $s \in (a,b)$, i.e.,

$$\langle e_i(s), e_j(s)\rangle = \delta_{ij} \ \ \forall\ s \in (a,b)$$

where δ_{ij} is the "Kronecker delta" symbol:

$$\delta_{ij} = \begin{cases} 0 & i \ne j \\ 1 & i = j\,. \end{cases} \tag{2.11}$$

Proof of the claim. We make use of the "Einstein summation convention":

$$\frac{d}{ds}e_i = \sum_{j=1}^{3} \Omega_i{}^j e_j = \Omega_i{}^j e_j$$

Let $g_{ij} = \langle e_i, e_j \rangle,\ g_{ij} = g_{ij}(s), 1 \le i, j \le 3$. Note,

$$\begin{aligned} g_{ij}(s_0) &= \langle e_i(s_0), e_j(s_0) \rangle \\ &= \delta_{ij} \end{aligned}$$

The g_{ij}'s satisfy a system of linear ODE's,

$$\begin{aligned} \frac{d}{ds}g_{ij} &= \frac{d}{ds}\langle e_i, e_j \rangle \\ &= \langle e_i', e_j \rangle + \langle e_i, e_j' \rangle \\ &= \langle \Omega_i{}^k e_k, e_j \rangle + \langle e_i, \Omega_j{}^\ell e_\ell \rangle \\ &= \Omega_i{}^k \langle e_k, e_j \rangle + \Omega_j{}^\ell \langle e_i, e_\ell \rangle \end{aligned}$$

Hence,

$$\frac{d}{ds}g_{ij} = \Omega_i{}^k g_{kj} + \Omega_j{}^\ell g_{i\ell}$$
$$\text{IC}:\ \ g_{ij}(s_0) = \delta_{ij}$$

Observe, $g_{ij} = \delta_{ij}$ is a solution to this system,

$$\begin{aligned} LHS &= \frac{d}{ds}\delta_{ij} = \frac{d}{ds}\text{const} = 0. \\ RHS &= \Omega_i{}^k \delta_{kj} + \Omega_j{}^\ell \delta_{i\ell} \\ &= \Omega_i{}^j + \Omega_j{}^i \\ &= 0 \text{ (skew symmetry!)}. \end{aligned}$$

But ODE theory guarantees a *unique* solution to this system. Therefore $g_{ij} = \delta_{ij}$ is *the* solution, and hence the claim follows.

How to define σ: Well, if $s \to \sigma(s)$ is a unit speed curve then

$$\sigma'(s) = T(s) \quad \Rightarrow \quad \sigma(s) = \sigma(s_0) + \int_{s_0}^{s} T(s)ds.$$

Hence, we *define* $s \to \sigma(s), s \in (a, b)$ by,

$$\sigma(s) = P_0 + \int_{s_0}^{s} e_1(s)ds$$

Claim. σ is unit speed, $\kappa = \overline{\kappa}$, $\tau = \overline{\tau}$, $T = e_1$, $N = e_2$, $B = e_3$.

Proof of the claim. We have,

$$\sigma' = \frac{d}{ds}(P_0 + \int_{s_0}^{s} e_1(s)ds) = e_1\,,$$

hence, $|\sigma'| = |e_1| = 1$, so σ is unit speed. Then we have,

$$T = \sigma' = e_1$$

$$\kappa = |T'| = |e_1'| = |\overline{\kappa} e_2| = \overline{\kappa}$$

$$N = \frac{T'}{\kappa} = \frac{e_1'}{\overline{\kappa}} = \frac{\overline{\kappa} e_2}{\overline{\kappa}} = e_2$$

$$B = T \times N = e_1 \times e_2 = e_3$$

$$B' = e_3' = -\overline{\tau} e_2 = -\overline{\tau} N \quad \Rightarrow$$

$$\tau = \overline{\tau}\,.$$

Additional Chapter 2 Exercises

1. Show that $\sigma(t) = (e^t \cos t, e^t \sin t, e^t)$ is a regular curve by computing its speed. Then reparameterize σ in terms of arc length.

2. Show that
$$\sigma(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}}\right)$$
is a unit speed curve, and compute its curvature.

3. (Continuation of Exercise 2.5.) If $\sigma : [0, \ell] \to \mathbb{R}^3$, $s \to \sigma(s)$, is a *convex* curve, then $\phi = \phi(s)$ is necessarily increasing. Hence $\int_0^\ell \kappa(s)ds = \triangle\phi = \phi(\ell) - \phi(0)$. If σ is a simple closed convex curve, conclude that $\int_0^\ell \kappa(s)ds = 2\pi$.

4. Use Proposition 2.7, p. 33, to compute κ, τ, T, N, B for $\sigma(t) = (\cosh t, \sinh t, t)$. (This curve is sometimes referred to as the hyperbolic helix.) Partial answer: $\kappa = \tau = \frac{1}{2}\mathrm{sech}^2 t$ (where $\mathrm{sech}\, t = \frac{1}{\cosh t}$).

5. Let $t \to \sigma(t)$ be a regular curve in $\mathbb{R}^3$. Suppose there is a point p in $\mathbb{R}^3$ such that $\sigma(t) - p$ is perpendicular to $\sigma'(t)$ for all t. Show that σ lies on sphere centered at p.

6. *Prescribed curvature in the plane.*

   (a) To construct a unit speed curve in the x-y with curvature $\kappa = \kappa(s)$ ($\kappa \geq 0$), let $\theta = \theta(s)$ be defined by, $\theta(s) = \int_0^s \kappa(u)\, du$. Then show that
   $$\sigma(s) = (\int_0^s \cos\theta(u)\, du, \int_0^s \sin\theta(u)\, du, 0)$$
   is a unit speed curve with curvature $\kappa = \kappa(s)$.

   (b) Use part (a) to construct a curve in the x-y plane with curvature $\kappa(s) = \frac{1}{1+s^2}$. (Ans.: $\sigma(s) = (\sinh^{-1}(s), \sqrt{1+s^2}, 0)$.)

7. (*Central force field.*) Consider the path of a particle $t \to \sigma(t)$ in space. Show that if the acceleration vector is always proportional to the position vector, $\sigma''(t) = f(t)\sigma(t)$ then the motion is in a plane. (Hint: Use Proposition 2.7(e).)

8. The rotation of a *rigid body* moving along a unit speed curve $s \to \sigma(s)$ is described by the *angular velocity* vector $\omega = \omega(s)$ determined by the equations, $T' = \omega \times T$, $N' = \omega \times N$, and $B' = \omega \times B$. Show that ω, in terms of T, N, and B, is given by $\omega = \tau T + \kappa B$. (Hint: write $\omega = aT + bN + cB$, and take cross products with T, N, and B to determine a, b , and c.) ω is known as the *Darboux* vector.

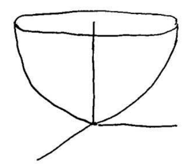
9. Let $\sigma : [0, L] \to \mathbb{R}^3$ be a unit speed *closed* curve, i.e. $\sigma(0) = \sigma(L)$. Show that if σ is contained in the ball of radius R centered at the origin then the integral of the curvature κ of σ satisfies, $\int_0^L \kappa ds \geq \frac{L}{R}$. (Hint differentiate $f(s) = -\langle \sigma'(s), \sigma(s) \rangle$ and then integrate.)

Chapter 3

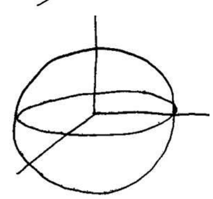
Surfaces

We all understand intuitively what a surface is. In calculus we encounter surfaces in several ways.

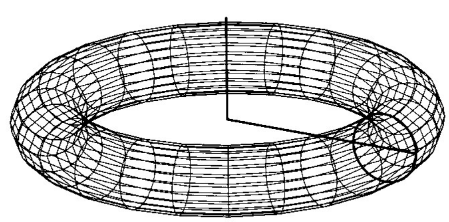
1. As graphs of functions of two variables, $z = f(x, y)$.
Ex. $z = x^2 + y^2$



2. As *level* surfaces of functions of three variables, $F(x, y,$
Ex. $x^2 + y^2 + z^2 = 1$



3. As surfaces of revolution.
Ex. Torus: surface of a doughnut.

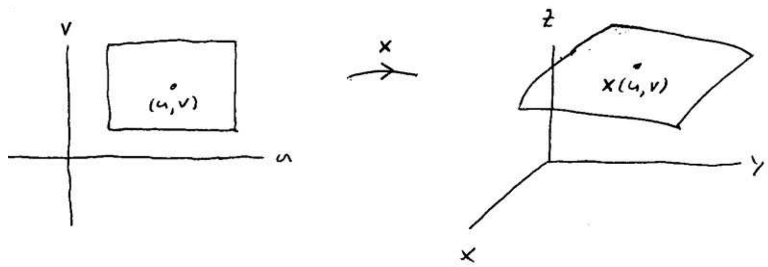


We shall be fairly precise about what we mean by a surface. Our definition will need to cover all these cases. The key is to describe surfaces parametrically. Very roughly speaking, a surface for us is going to be a subset of $\mathbb{R}^3$ which can be broken up into overlapping pieces such that each piece is described parametrically, i.e. described by a 2-parameter map.

Hence, the starting point is the notion of *parameterized surfaces.*

Definition. A smooth parameterized surface in $\mathbb{R}^3$ is a smooth map $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \to \mathbf{x}(u, v)$.

As (u, v) varies over U, $\mathbf{x}(u, v) \in \mathbb{R}^3$ traces out a "surface" in $\mathbb{R}^3$.



In terms of components, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$,

$$\mathbf{x} : \begin{array}{l} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{array} \qquad (u, v) \in U$$

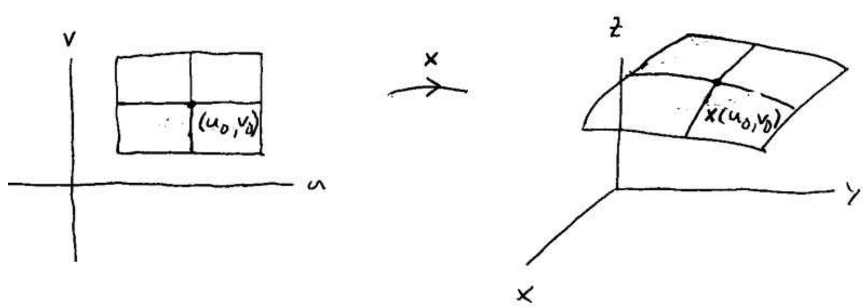
An effective way to see what gets traced out is to look at the "u-curves" and "v-curves".

(1) if v is held constant, $v = v_0$ and u varies,

$$u \to \mathbf{x}(u, v_0) \qquad \text{“}u - \text{curve}''$$

(2) if u is held constant, $u = u_0$ and v varies,

$$v \to \mathbf{x}(u_0, v) \qquad \text{“}v - \text{curve}''$$



One way to examine a parameterized surface is to plot many "coordinate" curves, u=const, v=const. This is how e.g., Mathematica plots parameterized surfaces.

Example. $\mathbf{x} : U \to \mathbb{R}^3$, $U = \{(u,v) : 0 < u < 2\pi, 0 < v < 3\}$, $\mathbf{x}(u,v) = (2\cos u, 2\sin u, v)$,

$$\mathbf{x} : \begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = v \end{array} \qquad 0 < u < 2\pi, \quad 0 < v < 3$$

For this example it is convenient to consider closed rectangle $\overline{U} : 0 \le u \le 2\pi$, $0 \le v \le 3$. We plot some u-curves and v-curves:

$$\underline{v = 0 :} \quad \begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = 0 \end{array} \qquad 0 \le u \le 2\pi \qquad \text{circle in } z = 0$$

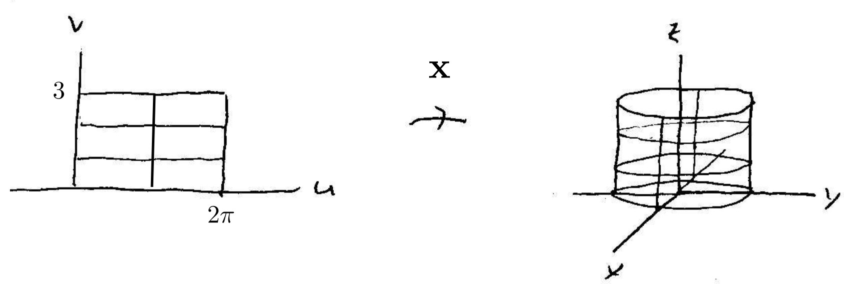
$$\underline{v = 1 :} \quad \begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = 1 \end{array} \qquad 0 \le u \le 2\pi \qquad \text{circle in } z = 1$$

etc.

$$\underline{u = 0 :} \quad \begin{array}{l} x = 2 \\ y = 0 \\ z = v \end{array} \qquad 0 \le v \le 3 \qquad \text{vertical line}$$

$$\underline{u = \pi/2 :} \quad \begin{array}{l} x = 0 \\ y = 2 \\ z = v \end{array} \qquad 0 \le v \le 3 \qquad \text{vertical line}$$

etc.



This parameterized surface describes a cylinder. Note that the coordinate functions satisfy:

$$x^2 + y^2 = 4, \qquad 0 \le z \le 3\,.$$

Note: On the original domain U, $\mathbf{x}$ is 1-1. We will restrict attention to parameterized surfaces $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ which are 1-1. Cylinder of radius $a : \mathbf{x}(u, v) = (a\cos u,\ a\sin u, v)$

<u>Coordinate Vector Fields</u>. Given a smooth surface,

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

we can differentiate wrt u and v,

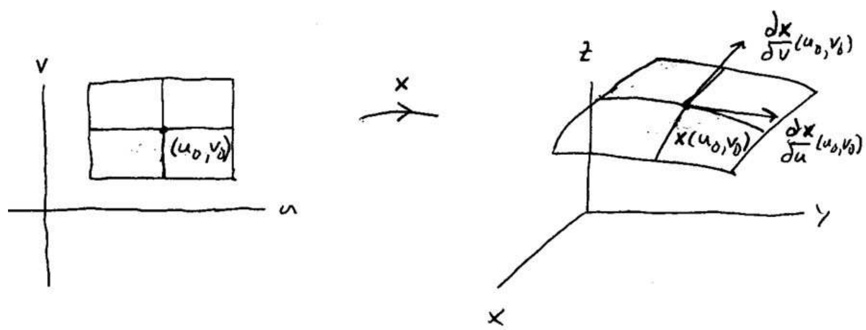
$$\frac{\partial \mathbf{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\frac{\partial \mathbf{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\,.$$

These partial derivatives have natural interpretations,

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial u}(u_0, v_0) &= \text{tangent vector to } u\text{-curve}\\ &\quad\ u \to \mathbf{x}(u, v_0) \text{ at } \mathbf{x}(u_0, v_0)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial v}(u_0, v_0) &= \text{tangent vector to } v\text{-curve}\\ &\quad\ v \to \mathbf{x}(u_0, v) \text{ at } \mathbf{x}(u_0, v_0)\end{aligned}$$



Hence,

$$\frac{\partial \mathbf{x}}{\partial u} = \text{velocity vector field to } u\text{-curves}$$

$$\frac{\partial \mathbf{x}}{\partial v} = \text{velocity vector field to } v\text{-curves.}$$

Remark. The coordinate curves $u = u_0,\ v = v_0$ lie in the surface. Hence the coordinate vectors $\frac{\partial \mathbf{x}}{\partial u}(u_0, v_0), \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$ are *tangent* vectors to the surface at $\mathbf{x}(u_0, v_0)$.

Standard Picture: Grid of horizontal and vertical lines in $U \subset \mathbb{R}^2$ gives rise to a grid of curves - the coordinate curves on $\mathbf{x}(U)$. This amounts to introducing coordinates on $\mathbf{x}(U)$.

Shorthand Notation: $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u},\ \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$.

Actually, to insure that the image of a parameterized surface $\mathbf{x}$ looks like a surface (i.e. smooth 2-dimensional object), we need a *regularity* condition, akin to the regularity condition for parameterized curves ($\sigma'(t) \neq 0$).

Example. $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3,\ \mathbf{x}(u, v) = (0, 0, 0)\ \forall (u, v)$. Image a single point! Note that $\frac{\partial \mathbf{x}}{\partial u} = \frac{\partial \mathbf{x}}{\partial v} = \mathbf{0}$.

Example. $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3,\ \mathbf{x}(u, v) = (\cos(u + v^2), \sin(u + v^2), 1)$

$$\mathbf{x} : \begin{array}{l} x = \cos(u + v^2) \\ y = \sin(u + v^2) \\ z = 1 \end{array}$$

Image: $x^2 + y^2 = 1, z = 1$, a circle!

Compute: $\frac{\partial \mathbf{x}}{\partial v} = -2v \frac{\partial \mathbf{x}}{\partial u}$, i.e. $\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}$ are linearly *dependent* (at every point).

To avoid this type of "degeneracy" we need to require that $\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}$ be linearly *independent*. There are several ways to characterize this independence.

Consider a parameterized surface, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$,

$$\mathbf{x} : \begin{array}{l} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{array}$$

$D\mathbf{x}$ = Jacobian matrix of $\mathbf{x}$, is the 3×2 matrix:

$$D\mathbf{x} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{bmatrix}$$

Recall,

$$\begin{aligned} \text{the rank of a matrix} &= \text{no. of linearly independent rows} \\ &= \text{no. of linearly independent columns} \end{aligned}$$

Proposition 3.1. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface. Then the following conditions are equivalent.*

(1) $D\mathbf{x}$ has rank 2.

(2) $\dfrac{\partial \mathbf{x}}{\partial u}, \dfrac{\partial \mathbf{x}}{\partial v}$ are linearly independent.

(3) $\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v} \neq 0$.

Proof:

$$\begin{aligned} D\mathbf{x} \text{ has rank } 2 &\iff \text{columns lin. indep.} \\[1ex] &\iff \frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v} \text{ lin. indep.} \\[1ex] &\iff \text{one is } \textit{not} \text{ a multiple of the other} \\[1ex] &\iff \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0. \end{aligned}$$

Definition. We introduce the following terminology.

1. A *regular (parameterized) surface* in $\mathbb{R}^3$ is a smooth parameterized surface $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v} \neq 0$ for all $(u, v) \in U$.

2. A *coordinate patch* is a regular surface $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ which is one-to-one.

Remark. Essentially (i.e., apart from an additonal technical condition discussed later), a surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is covered by coordinate patches.

If the regularity condition is not satisfied, the image of $\mathbf{x}$ can degenerate to a point, or curve – or something that does not look like a smooth surface (surface with "folds" or "cusps"). If, however, the regularity condition is satisfied, then the image of $\mathbf{x}$ will look like a smooth surface. This is made precise in the following proposition.

Proposition 3.2. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface. Then for each $(u_0, v_0) \in U$ there is a neighborhood $V \subset U$ of (u_0, v_0) such that the image $\mathbf{x}(V) \subset \mathbb{R}^3$ coincides with the graph of an equation of the form,*

$$z = f(x, y) \quad \text{or} \quad y = g(x, z) \quad \text{or} \quad x = h(y, z)\,,$$

where f, g, h are smooth functions of two variables.

Proof. The proof is a nice application of the Inverse Function Theorem. We have $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, and

$$\frac{\partial \mathbf{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right), \quad \frac{\partial \mathbf{x}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \\ &= \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \mathbf{k}\,. \end{aligned}$$

Since, by regularity, $\dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v} \neq 0$ at (u_0, v_0), one of the components must be nonzero, say,

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0 \quad \text{at } (u_0, v_0)\,.$$

Now, consider the map $\Phi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by, $\Phi(u, v) = (x(u, v), y(u, v))$,

$$\Phi : \begin{array}{l} x = x(u, v) \\ y = y(u, v) \end{array}$$

Φ has Jacobian matrix,

$$D\Phi = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

Hence $\det D\Phi \neq 0$ at (u_0, v_0), i.e. $D\Phi$ is nonsingular at (u_0, v_0). By the IFT there exists a neighborhood V of (u_0, v_0) such that $W = \Phi(V)$ is open in $\mathbb{R}^2$ and $\Phi^{-1} : W \subset \mathbb{R}^2 \to V \subset \mathbb{R}^2$ is smooth. In terms of components, $\Phi^{-1}(x, y) = (u(x, y), v(x, y))$,

$$\Phi^{-1} : \begin{array}{l} u = u(x, y) \\ v = v(x, y) \end{array}, (x, y) \in W$$

Now, let $f = z \circ \Phi^{-1}$, $f : W \subset \mathbb{R}^2 \to \mathbb{R}$,

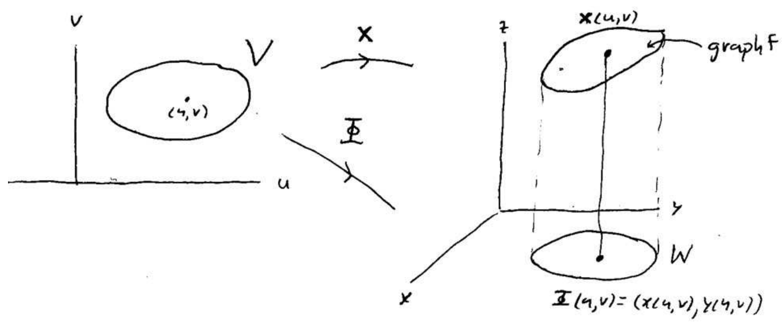
$$f(x, y) = z(\Phi^{-1}(x, y)) = z(u(x, y), v(x, y)).$$

The *graph* of f is the set of points in $\mathbb{R}^3$,

$$\text{graph}\, f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in W\}$$

We now observe that $\mathbf{x}(V) = \text{graph} f$ (see figure next page): $f = z \circ \Phi^{-1} \Rightarrow z = f \circ \Phi$, hence $z(u, v) = f(\Phi(u, v)) = f(x(u, v), y(u, v))$. Thus,

$$\begin{aligned} \mathbf{x}(u, v) &= (x(u, v), y(u, v), z(u, v)) \\ &= (x(u, v), y(u, v), f(x(u, y), y(u, v)) \in \text{graph}\, f. \end{aligned}$$



Some Parameterized Surfaces

1. Graphs of functions of two variables, $z = f(x, y)$, $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ smooth function,

$$\text{graph}\, f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in U\}.$$

For example: For $f(x, y) = x^2 + y^2$, graph f is the set of all points (x, y, z) such that $z = x^2 + y^2$.

There is a standard way to parameterize such graphs: $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (u, v, f(u, v))$,

$$\mathbf{x} : \begin{array}{l} x = u \\ y = v \\ z = f(u, v) \end{array} \qquad (u, v) \in U\,.$$

$\mathbf{x}$ is a regular surface (in fact, a coordinate patch). One needs to checl the regularity condition:

$$\frac{\partial \mathbf{x}}{\partial u} = \left(1, 0, \frac{\partial f}{\partial u}\right), \quad \frac{\partial \mathbf{x}}{\partial v} = \left(0, 1, \frac{\partial f}{\partial v}\right)$$
$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) \neq 0\,.$$

$\mathbf{x}$ is called the *Monge patch* associated to f.

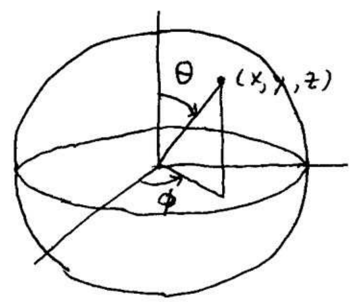
Example. $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sqrt{1-x^2-y^2}$, $U = \{(x,y) : x^2+y^2 < 1\}$, graph $f : z = \sqrt{1-x^2-y^2}$, a hemisphere. Associated Monge patch:

$$\mathbf{x} : \begin{array}{l} x = u \\ y = v \\ z = \sqrt{1-u^2-v^2} \end{array} ,$$

i.e., $\mathbf{x}(u,v) = (u, v, \sqrt{1-u^2-v^2})$, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$.

2. Geographical Coordinates on a sphere of radius R.

$$S_R^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$$



$\theta =$ colatitude, $0 \le \theta \le \pi$

$\phi =$ longitude, $0 \le \phi \le \pi$.

By spherical coordinates,

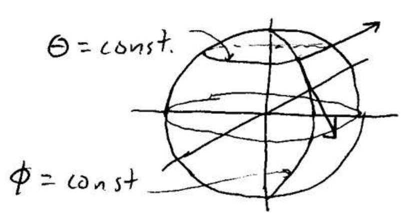
$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta\,. \end{aligned}$$

Let $U = \{(\theta,\phi) : 0 < \theta < \pi, \;\; 0 < \phi < 2\pi\}$. Define $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by,

$$\mathbf{x}(\theta,\phi) = (R\sin\theta\cos\phi, \;\; R\sin\theta\sin\phi, \;\; R\cos\theta)\,.$$

$\mathbf{x}$ is clearly a smooth parameterized surface.

Coordinate curves:



θ-curves: $\phi =$ const – longitudes (meridians)
ϕ-curves: $\theta =$ const – circles of latitude

Coordinate vector fields:

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \theta} &= (R\cos\theta\cos\phi, \quad R\cos\theta\sin\phi, -R\sin\theta) \\ \frac{\partial \mathbf{x}}{\partial \phi} &= (-R\sin\theta\sin\phi, \quad R\sin\theta\cos\phi, \quad 0)\,.\end{aligned}$$

E.g., at $(\theta, \phi) = (\pi/2, \ \ \pi/2)$,

$$\frac{\partial \mathbf{x}}{\partial \theta} = (0, 0, -R), \qquad \frac{\partial \mathbf{x}}{\partial \phi} = (-R, 0, 0)$$

EXERCISE 3.1. Show by computation that $\left|\frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi}\right| = R^2 \sin\theta > 0$, $0 < \theta < \pi$.

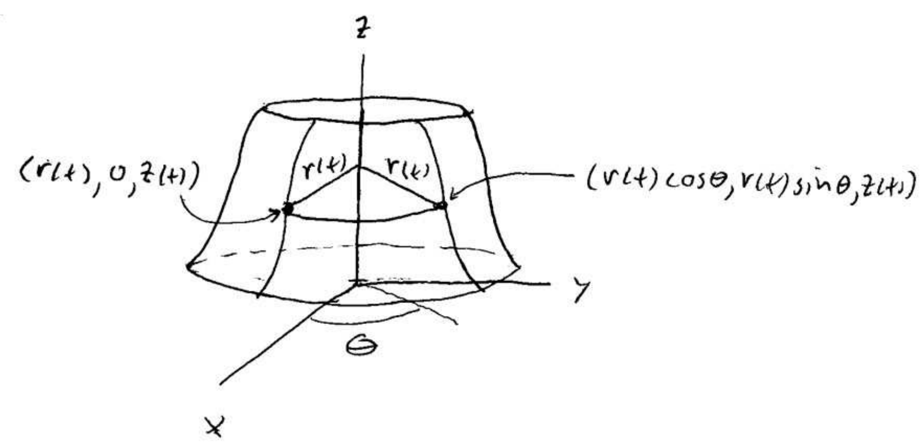
Hence, $\mathbf{x}$ is regular surface (in fact, a coordinate patch).

3. Surfaces of revolution.

Consider a regular curve σ in the x-z plane, $\sigma(t) = (r(t), 0, z(t))$, i.e,

$$\sigma : \begin{array}{l} x = r(t) \\ y = 0 \\ z = z(t) \end{array} \qquad a < t < b\,.$$

(Assume σ does not meet the z-axis.) Now rotate σ about the z-axis to generate a surface of revolution:

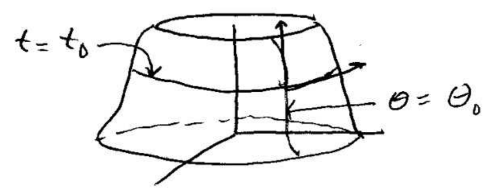


Parameterize as follows: Let $U = \{(t, \theta) : a < t < b. -\pi < \theta < \pi\}$. Define $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by,

$$\mathbf{x}(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

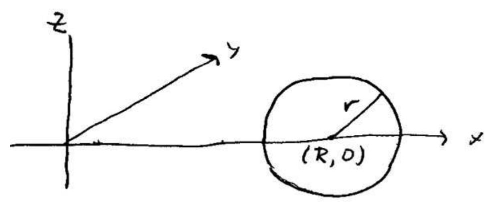
This gives a parametric description of the surface of revolution; t measures position along σ and θ measure how far σ has been rotated.

t-curves: θ=const, longitudes (meridians)
θ-curves: t = const, circles of latitude (parallels).



EXERCISE 3.2. Show that $\mathbf{x}$ as defined above is a regular surface (in fact a coordinate patch provided σ is 1-1).

EXERCISE 3.3. Rotate the circle pictured below about the z-axis to obtain a torus.



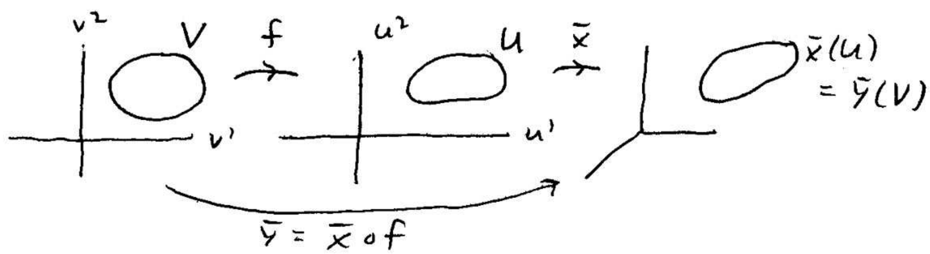
Show that the torus is parameterized by the following map: $U = (0, 2\pi) \times (-\pi, \pi)$,
$\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathbf{x}(t, \theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t).$$

Hint: Parameterize the circle appropriately.

Reparameterizations.

Definition. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface. Let $f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$ be a diffeomorphism. Then $\mathbf{y} = \mathbf{x} \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a reparameterization.



Proposition 3.3. *Given a regular surface* $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *and a diffeomorphism* $f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$*, the map* $\mathbf{y} = \mathbf{x} \circ f : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ *is a regular surface.*

Proof. $\mathbf{y} = \mathbf{x} \circ f$ is smooth. We need to show that $\mathbf{y}$ satisfies the regularity condition. To do this we first show how the two sets of coordinate vectors $\left\{ \dfrac{\partial \mathbf{x}}{\partial u^i} \right\}$, $\left\{ \dfrac{\partial \mathbf{y}}{\partial v^i} \right\}$ are related. Some notation:

$$f : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$$

$$f(v_1, v_2) = (u^1, u^2) = (f^1(v^1, v^2), f^2(v^1, v^2))$$

$$f : \begin{array}{l} u^1 = f^1(v^1, v^2) \\ u^2 = f^2(v^1, v^2) \end{array}$$

$$Df = \left[\frac{\partial u^i}{\partial v^k} \right]_{2\times 2}$$

Then,

$$\begin{aligned} \mathbf{y}(v^1, v^2) &= \mathbf{x} \circ f(v^1, v^2) = \mathbf{x}(f(v^1, v^2)) \\ &= \mathbf{x}\,(\underbrace{f^1(v^1, v^2)}_{u^1}, \underbrace{f^2(v^1, v^2))}_{u^2} \end{aligned}$$

i.e.

$$\mathbf{y} = \mathbf{x}(u^1, u^2) \quad \text{where } f : \begin{array}{l} u^1 = f^1(v^1, v^2) \\ u^2 = f^2(v^1, v^2) \end{array} .$$

Hence, by the chain rule,

$$\begin{aligned}
\frac{\partial \mathbf{y}}{\partial v^k} &= \frac{\partial \mathbf{x}}{\partial u^1}\frac{\partial u^1}{\partial v^k} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{\partial u^2}{\partial v^k}, \quad k = 1, 2 \\
&= \sum_{j=1}^{2} \frac{\partial \mathbf{x}}{\partial u^j}\frac{\partial u^j}{\partial v^k} \\
\frac{\partial \mathbf{y}}{\partial v^k} &= \sum_{j} \frac{\partial u^j}{\partial v^k}\frac{\partial \mathbf{x}}{\partial u^j}, \qquad\qquad k = 1, 2.
\end{aligned}$$

EXERCISE 3.4. Show that,

$$\begin{aligned}
\frac{\partial \mathbf{y}}{\partial v^1} \times \frac{\partial \mathbf{y}}{\partial v^2} &= \det Df \cdot \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \\
&= \frac{\partial(u^1, u^2)}{\partial(v^1, v^2)} \cdot \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \quad (\neq 0)
\end{aligned}$$

Hence, $\mathbf{y}$ is regular if $\mathbf{x}$ is.

Terminology: The reparameterization map f is called a *coordinate transformation*, and describes a change of coordinates on the surface.

Surfaces (at last!).

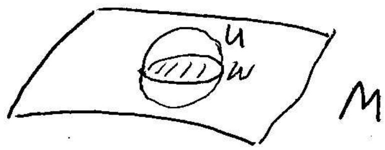
We now want to make the transition from the notion of a *parameterized* surface to that of a *surface*. A (regular) surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is covered by coordinate patches, subject to some additional conditions described below. The sphere, for example,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

will, by our definition, be a surface (as it should be!). It can be covered by several coordinate patches – but not by a single coordinate patch.

Before giving the "official" definition of a regular surface, we need to make a couple of comments.

1. Any subset M of $\mathbb{R}^3$ inherits a natural collection of 'open sets': We say that $W \subset M$ is *open* in M provided $W = U \cap M$ for some open set U in $\mathbb{R}^3$. (See figure on next page.)



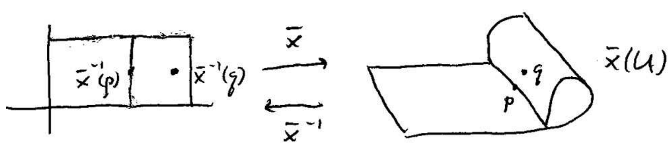
2. Consider a *coordinate patch*, i.e. a 1-1 regular parameterized surface, $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Then, in particular,

$$\mathbf{x} : U \to \mathbf{x}(U)$$

is a continuous, 1-1 and onto map. Hence we can consider the inverse,

$$\mathbf{x}^{-1} : x(U) \to U\,.$$

We point out that it's possible for the inverse to be *not* continuous:



The points p, q are close, as points in $\mathbb{R}^3$, but $\mathbf{x}^{-1}(p), \mathbf{x}^{-1}(q)$ are not close.

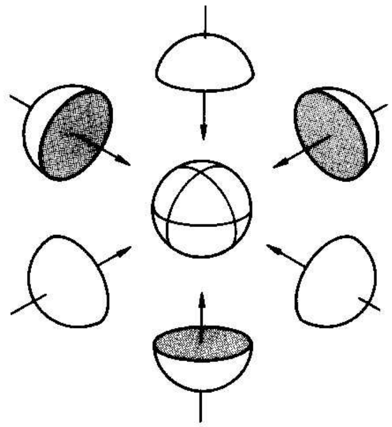
Definition. A subset $M \subset \mathbb{R}^3$ is a *(regular) surface* provided each point of M is contained in a coordinate patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that (i) $\mathbf{x}(\mathbf{U})$ is open in M and (ii) $\mathbf{x}^{-1} : x(U) \to U$ is continuous.

We refer to coordinate patches with the additional properties in the defintion as *proper patches.* Thus, $M \subset \mathbb{R}^3$ is a regular surface iff M can be *covered* by proper patches.

Example. Consider the sphere,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

In this example the sphere is covered by six proper patches: $z^+, z^-, y^+, y^-, x^+, x^-$, each a parameterized hemisphere.



z^+: *upper hemisphere*: $z = \sqrt{1-x^2-y^2}$, with domain $D : x^2+y^2 < 1$. Associated Monge patch: $z^+ : U \to \mathbb{R}^3$, $U = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\}$,

$$z^+ : \begin{array}{l} x = u \\ y = v \\ z = \sqrt{1-u^2-v^2}, \end{array}$$

i.e. $\quad z^+(u,v) = (u, v, \sqrt{1-u^2-v^2})$.

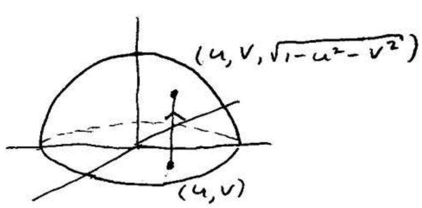
Claim. z^+ is a proper patch in S^2.

(1) z^+ is a coordinate patch (Monge patch)

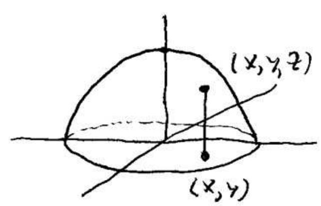
(2) $z^+(U)$ is an open subset of S^2:

$$\begin{aligned} z^+(U) &= \{(x,y,z) \in S^2 : z > 0\} \\ &= S^2 \cap \underbrace{\{z > 0\}}_{\text{open in } \mathbb{R}^3} . \end{aligned}$$

(3) $(z^+)^{-1} : z^+(U) \to U$ is continuous:



z^+



$(z^+)^{-1}$

$(z^+)^{-1}(x,y,z) = (x,y)$ – projection onto the first two coordinates, which is continuous.

z^-: lower hemisphere: $z = -\sqrt{1-x^2-y^2}$; associated Monge patch: $z^- : U \to \mathbb{R}^3, z^-(u,v) = (u,v,-\sqrt{1-u^2-v^2})$.

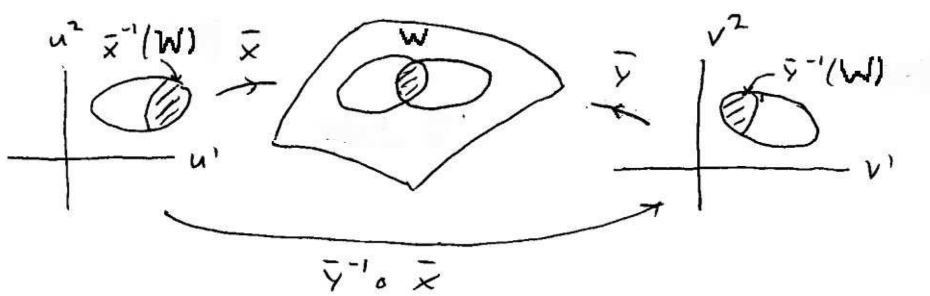
Other hemispheres:

y^+ = Monge patch associated with hemisphere $S^2 \cap \{y > 0\}$ $\quad (y = \sqrt{1-x^2-z^2})$
y^- = – – –" – – – $S^2 \cap \{y < 0\}$
x^+ = – – –" – – – $S^2 \cap \{x > 0\}$
x^- = – – –" – – – $S^2 \cap \{x < 0\}$.

Proposition 3.4 (Smooth overlap property)**.** *Let M be a surface. Let $\mathbf{x} : U \to \mathbb{R}^3$ and $\mathbf{y} : V \to \mathbb{R}^3$ be two proper patches in M which overlap, $W := \mathbf{x}(\mathbf{U}) \cap \mathbf{y}(\mathbf{V}) \neq \emptyset$ Then,*

$$\mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \subset \mathbb{R}^2 \to \mathbf{y}^{-1}(W) \subset \mathbb{R}^2$$

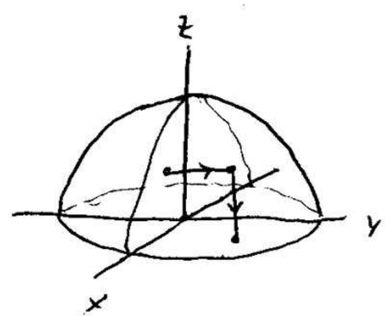
is a diffeomorphism (i.e., is smooth with smooth inverse).



Proof. Inverse function theorem! (See [1], p. 70, Propisition 1.)

Example. In sphere example consider the overlapping patches $z^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\quad z^+(u,v) = (u,v,\sqrt{1-u^2-v^2})$ and $y^+ : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $y^+(u,v) = (u,\sqrt{1-u^2-v^2},v)$. Observe, $y^+(U) = S^2 \cap \{y > 0\}$ and $z^+(U) = S^2 \cap \{z > 0\}$. Then,

$$W := y^+(U) \cap z^+(U) = S^2 \cap \{y > 0\} \cap \{z > 0\} \neq \emptyset\,.$$



Consider $(z^+)^{-1} \circ y^+ : (y^+)^{-1}(W) \to (z^+)^{-1}(W)$. Note, $(y^+)^{-1}(W) =$ half-disk $= U \cap \{v > 0\}$. Now,

$$\begin{aligned} y^+(u,v) &= (u, \sqrt{1-u^2-v^2}, v) \\ (z^+)^{-1}(x,y,z) &= (x,y) \end{aligned}$$

and hence

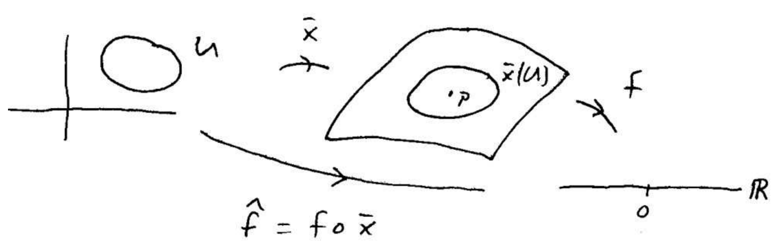
$$\begin{aligned} (z^+)^{-1} \circ y^+(u,v) &= (z^+)^{-1}(y^+(u,v)) = (z^+)^{-1}(u, \sqrt{1-u^2-v^2}, v) \\ &= (u, \sqrt{1-u^2-v^2})\,, \end{aligned}$$

which is smooth on $U \cap \{v > 0\}$.

The smooth overlap property is the key ingredient used to generalize the notion of surfaces in $\mathbb{R}^3$ to *differentiable manifolds*. That this property holds for surfaces is important. For example, it is used to show that certain properties which are defined in terms of coordinate *charts* (proper charts), don't really depend on the specific coordinate charts chosen. We give an illustration.

Consider a function $f : M \to \mathbb{R}$, where M is a surface. What does it mean for f to be smooth?

Definition. $f : M \to \mathbb{R}$ is *smooth* provided for each $p \in M$ there exists a proper patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ in M containing p ($p \in \mathbf{x}(U) \subset M$) such that $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth.



We say that $\hat{f}$ is f expressed in coordinates. Thus, f is smooth provided there exists a collection of charts covering M such that each coordinate expression $\hat{f}$ is smooth.

This definition of smoothness does not depend on the particular choice of proper charts covering M.

EXERCISE 3.5. If $\mathbf{x}$ and $\mathbf{y}$ are *any* two overlapping proper patches in M then on the overlap, $f \circ \mathbf{x}$ is smooth iff $f \circ \mathbf{y}$. (*Hint*: Smooth overlap property.)

The following proposition identifies a large and important class of surfaces.

Proposition 3.5 (The inverse image theorem)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Consider the level set*

$$M = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}$$

If $\nabla f = (f_x, f_y, f_z) \neq 0$ at each point of M then M is a surface.

Example. The sphere.

$$\begin{aligned} S^2 &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \\ &= \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}, \end{aligned}$$

where $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Now,

$$\nabla f = (2x, 2y, 2z),$$

and so $\nabla f \neq 0$ except at $(x, y, z) = (0, 0, 0) \notin S^2$. Hence, $\nabla f \neq 0$ at each point of S^2. Therefore S^2 is a surface.

Example. Double Cone.

$$\begin{aligned} M &= \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\} \\ &= \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\}, \end{aligned} ,$$

where $f(x, y, z) = x^2 + y^2 - z^2$. Then,

$$\nabla f = (2x, 2y, 2z) \neq 0$$

except at $(0,0,0)$. But this time $(0,0,0) \in M$. So the proposition doesn't guarantee that M is a surface, and in fact it is not. The origin is not contained in a proper patch. In general, however, away from points where the gradient vanishes we do get a surface.

Remarks

1. $Df = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \sim (f_x, f_y, f_z) = \nabla f$. I.e. the gradient of f essentially corresponds to the Jacobian of f.

2. Because $M = f^{-1}(0)$. For this reason, the proposition is often referred to as the *inverse image theorem.*

Sketch of proof. Uses the inverse function theorem (actually the implicit function theorem, which is a consequence of the IFT; see [1], p. 59, Prop. 2 for complete details).

We want to show M is covered by proper patches. Choose $p_0 = (x_0, y_0, z_0) \in M$; $\nabla f|_{p_o} \neq 0$. Suppose then that $\dfrac{\partial f}{\partial z}(p_0) \neq 0$.

$$M: \quad f(x,y,z) = 0 \qquad (*)$$

By the IFT, near $p_0 = (x_0, y_0, z_0)$, $(*)$ can be solved smoothly for z in terms of x and y,

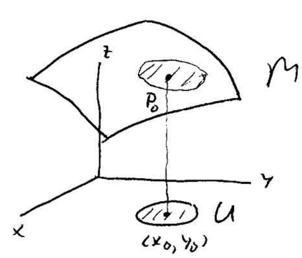
$$z = h(x,y)\,,$$

i.e., there exists a nbd U of (x_0, y_0) and a smooth function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $(x, y, h(x,y))$ satisfies $(*)$ for all (x,y) in U,

$$f(x, y, h(x,y)) = 0 \quad \forall (x,y) \in U\,.$$

Hence,

$$(x, y, h(x,y)) \in M \qquad \text{for all } (x,y) \in U.$$



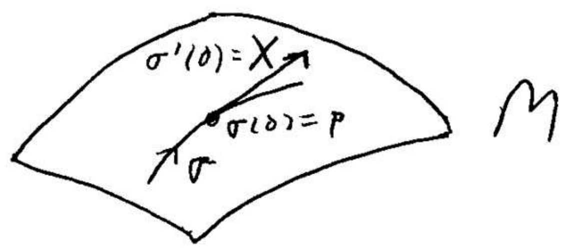
Now consider the Monge patch associated to h, $\mathbf{x} : U \to \mathbb{R}^3$,

$$\mathbf{x}(u,v) = (u, v, h(u,v))\,.$$

Then $\mathbf{x}$ is a proper patch in M which contains p_0.

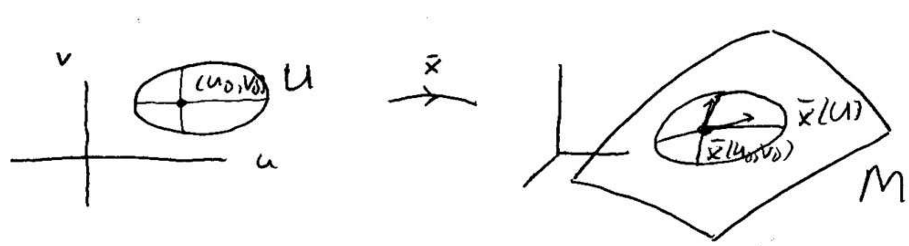
Tangent Vectors to a Surface.

Definition. Let M be a surface, and $p \in M$. X is a tangent vector to M at p provided X is the velocity vector at p of some smooth curve σ which lies in M, i.e. provided there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3$ such that $\sigma(0) = p$ and $\sigma'(0) = X$.



This definition is independent of coordinate patches – it is a coordinate free concept. But for computational purposes it's convenient to introduce coordinates.

Let $\mathbf{x} : U \to M \subset \mathbb{R}^3$ be a proper patch in M which contains p, $p = \mathbf{x}(u_0, v_0)$.



Observe that $\dfrac{\partial \mathbf{x}}{\partial u}(u_0, v_0)$ and $\dfrac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$ are tangent vectors to M at $p = \mathbf{x}(u_0, v_0)$: according to the definition:

$$\mathbf{x}_u(u_0, v_0) = \text{velocity vector to } u \to \mathbf{x}(u, v_0) \text{ at } \mathbf{x}(u_0, v_0)\text{, and,}$$

$$\mathbf{x}_v(u_0, v_0) = \text{velocity vector to } v \to \mathbf{x}(u_0, v) \text{ at } \mathbf{x}(u_0, v_0)\ .$$

Notation/Terminology.

$$\begin{aligned} T_p\mathbb{R}^3 &:= \text{tangent space of } \mathbb{R}^3 \text{ at } p \\ &= \text{set of all vectors in } \mathbb{R}^3 \text{ based at } p. \end{aligned}$$

$T_p\mathbb{R}^3$ is a 3-dimensional vector space. For M a surface, $p \in M$,

$$\begin{aligned} T_pM &:= \text{tangent space of } M \text{ at } p \\ &= \text{set of all tangent vectors to } M \text{ at } p. \end{aligned}$$

In the following proposition we show that T_pM is a 2-dimensional subspace of $T_p\mathbb{R}^3$ spanned by $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.

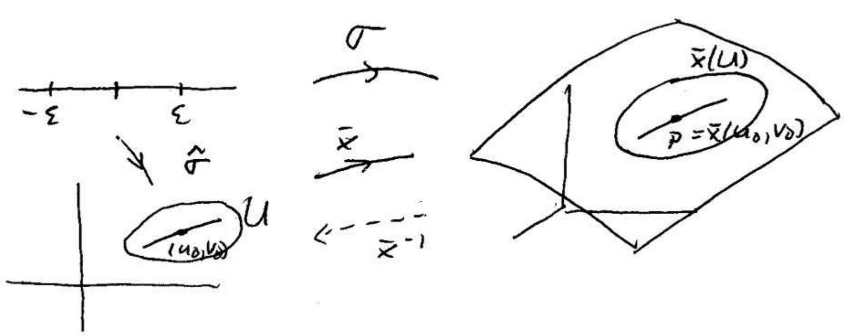
Proposition 3.6. *Let M be a surface, $p \in M$. Let $\mathbf{x} : U \to M \subset \mathbb{R}^3$ be a proper patch in M containing p, $p = \mathbf{x}(u_0, v_0)$. Then T_pM is a 2-dimensional vector space, in fact it is the 2-dimensional vector subspace of $T_p\mathbb{R}^3$ spanned by $\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\}$,*

$$\begin{aligned} T_pM &= \operatorname{span}\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\} \\ &= \{A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0) : A, B \in \mathbb{R}\} \end{aligned}$$

Proof:
$T_pM \subset \operatorname{span}\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\}$: Let $X \in T_pM$. Then there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^3$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Without loss of generality, by taking ϵ sufficiently small, $\sigma \subset \mathbf{x}(U)$.

Key observation: σ can be represented in a certain manner in terms of coordinates; we will use this representation over and over.

Let $\hat{\sigma} = \mathbf{x}^{-1} \circ \sigma$:



$\hat\sigma : (-\epsilon, \epsilon) \to U \subset \mathbb{R}^2$, and in terms of components, $\hat\sigma(t) = (u(t), v(t))$, $t \in (-\epsilon, \epsilon)$,

$$\hat\sigma : \begin{array}{l} u = u(t) \\ v = v(t) \end{array} \qquad -\epsilon < t < \epsilon\,. \tag{3.1}$$

$\hat\sigma(0) = \mathbf{x}^{-1}(\sigma(0)) = \mathbf{x}^{-1}(p) = (u_0, v_0)$. Using the IFT, it can be shown that $\hat\sigma$ is a smooth curve in $\mathbb{R}^2$, that is, $u = u(t)$ and $v = v(t)$ are smooth functions.

Now, $\hat\sigma = \mathbf{x}^{-1} \circ \sigma \Rightarrow \quad \sigma = \mathbf{x} \circ \hat\sigma \Rightarrow \sigma(t) = \mathbf{x}(\hat\sigma(t))$, i.e.

$$\sigma(t) = \mathbf{x}(u(t), v(t))\,, \qquad t \in (-\epsilon, \epsilon)\,. \tag{3.2}$$

Remark. We say that $\hat\sigma$ is the *coordinate representation* of σ; $\hat\sigma$ is just σ expressed in coordinates.

Returning to the proof, by the chain rule,

$$\frac{d\sigma}{dt} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt},$$

or, rewriting slightly,

$$\sigma'(t) = u'(t)\mathbf{x}_u(u(t), v(t)) + v'(t)\mathbf{x}_v(u(t), v(t)),$$

and setting $t = 0$, we obtain,

$$\sigma'(0) = u'(0)\mathbf{x}_u(u_0, v_0) + v'(0)\mathbf{x}_v(u_0, v_0)\,,$$

and thus,

$$X = A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0)\,,$$

where $A = u'(0), B = v'(0)$, as was to be shown.

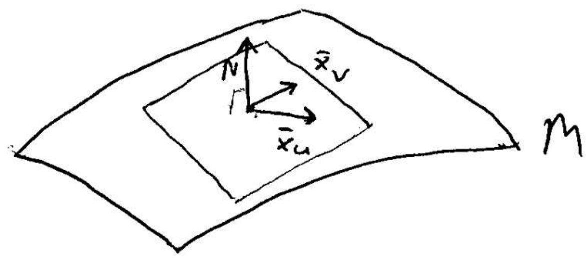
$\underline{\text{span}\,\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\} \subset T_pM}$: One must show that a vector of the form,

$$A\mathbf{x}_u(u_0, v_0) + B\mathbf{x}_v(u_0, v_0)$$

for any $A, B \in \mathbb{R}$, is the velocity vector of a curve σ in M passing through p.

EXERCISE 3.6. Show this. Hint: Let $\sigma = \mathbf{x} \circ \hat{\sigma}$ where $\hat{\sigma}$ is the parameterized line, $\hat{\sigma}(t) = (At+u_0, Bt+v_0)$. Then, $\sigma(t) = \mathbf{x}(\hat{\sigma}(t)) = \mathbf{x}(At+u_0, Bt+v_0)$, and apply the chain rule.

<u>Tangent plane to M at p:</u>



Let $\mathbf{x}$ be a proper patch in M containing $p = \mathbf{x}(u_0, v_0)$. Then the

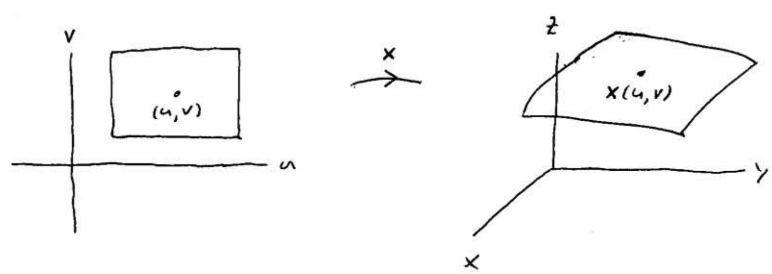
$$\begin{aligned} \text{tangent plane to } M \text{ at } p &= \text{plane through } p \text{ spanned by} \\ &\quad\ \mathbf{x}_u(u_0, v_0) \text{ and } \mathbf{x}_v(u_0, v_0) \\ &= \text{plane through } p \text{ perpendicular} \\ &\quad\ \text{to } N = \mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0). \end{aligned}$$

Equation of tangent plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\,,$$

where $N = (a, b, c)$ and $p = \mathbf{x}(u_0, v_0) = (x_0, y_0, z_0)$.

<u>Unit normal vector field</u> associated to a proper patch $\mathbf{x} : U \to M \subset \mathbb{R}^3$:



$\mathbf{n} = \mathbf{n}(u, v)$, $\mathbf{n}(u, v) \in T_{\mathbf{x}(u,v)}\mathbb{R}^3$, $\mathbf{n}(u, v) \perp M$.

Remark. The unit normal field is *unique up to sign.*

Example. Compute the unit normal field to the surface $z = x^2 + y^2$ with respect to the associated Monge patch.

We have, $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x}(u, v) = (u, v, u^2 + v^2)$, $\mathbf{x}_u = (1, 0, 2u), \mathbf{x}_v = (0, 1, 2v)$, and so,

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2u \end{bmatrix} = (-2u, -2v, 1)$$

Hence,

$$\begin{aligned} \mathbf{n} &= \frac{(-2u, -2v, 1)}{|(-2u, -2v, 1)|} \\ \mathbf{n}(u, v) &= \frac{(-2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}}. \end{aligned}$$

EXERCISE 3.7. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let M=graph f. M is a smooth surface covered by a single patch - the associated Monge patch $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{x}(u, v) = (u, v, f(u, v))$. Show that the unit normal vector field to M wrt $\mathbf{x}$ is given by,

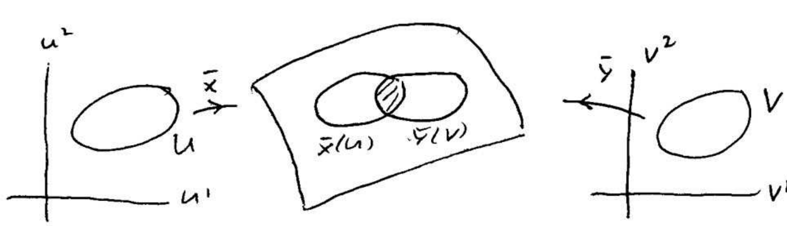
$$\mathbf{n} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

where $f_u = \dfrac{\partial f}{\partial u}$ and $f_v = \dfrac{\partial f}{\partial v}$.

<u>Some Tensor Analysis</u>

Consider overlapping patches,

$$\begin{aligned} &\mathbf{x} : U \to M \subset \mathbb{R}^3 \qquad && \mathbf{x} = \mathbf{x}(u^1, u^2) \\ &\mathbf{y} : V \to M \subset \mathbb{R}^3 \qquad && \mathbf{y} = \mathbf{y}(v^1, v^2). \end{aligned}$$



Let $p \in \mathbf{x}(U) \cap \mathbf{y}(V)$. T_pM has the two different bases at p: $\{\mathbf{x}_1, \mathbf{x}_2\}$, $\{\mathbf{y}_1, \mathbf{y}_2\}$ where we are using the shorthand, $\mathbf{x}_1 = \dfrac{\partial \mathbf{x}}{\partial u^1}, \mathbf{x}_2 = \dfrac{\partial \mathbf{x}}{\partial u^2}, \mathbf{y}_1 = \dfrac{\partial \mathbf{y}}{\partial v^1}, \mathbf{y}_2 = \dfrac{\partial \mathbf{u}}{\partial v^2}$.

Let $X \in T_pM$. X can be expressed in two different ways,

$$\begin{aligned} X &= \sum_{i=1}^{2} X^i \mathbf{x}_i \\ &= \sum_{k=1}^{2} \tilde{X}^k \mathbf{y}_k \end{aligned}$$

Classical tensor analysis is concerned with questions like the following: How are the components X^i and $\tilde{X}^k$ with respect to the two different bases related? We now consider this.

By the smooth overlap property, $f = \mathbf{y}^{-1} \circ \mathbf{x}$ is a diffeomorphism on the overlap. We have, $f : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$, where $W = \mathbf{x}(U) \cap \mathbf{x}(V)$, and $f(u^1, u^2) = (v^1, v^2) = (f^1(u^1, u^2), f^2(u^1, u^2))$,

$$f : \begin{array}{l} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{array} ,$$

i.e., f is the change of coordinates map; v^1 and v^2 depend smoothly on u^1 and u^2. On the overlap we have, $\mathbf{x} = \mathbf{y} \circ f$, and hence, $\mathbf{x}(u^1, u^2) = \mathbf{y}(v^1, v^2)$, where v^1, v^2 depend on u^1, u^2 as above.

EXERCISE 3.8. (1) Use the chain rule to show,

$$\mathbf{x}_i = \sum_k \frac{\partial v^k}{\partial u^i} \mathbf{y}_k$$

(Note: This is essentially the same as the computation on p. 53, but with the role of $\mathbf{x}$ and $\mathbf{y}$ reversed from that here).

(2) Use (1) to show,

$$\tilde{X}^k = \sum_i \frac{\partial v^k}{\partial u^i} X^i, \quad k = 1, 2.$$

(3) Show (2) implies

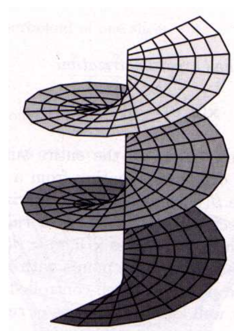
$$\begin{bmatrix} \tilde{X}^1 \\ \tilde{X}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial v^k}{\partial u^i} \end{bmatrix}}_{Df} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}$$

Additional Chapter 3 Exercises

1. The helicoid is the surface swept out by a rotating horizontal line as it rises along the z-axis (see the figure below). It can be described by the parameterized surface $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathbf{x}(u, v) = (au \cos v, au \sin v, bv)\,,$$

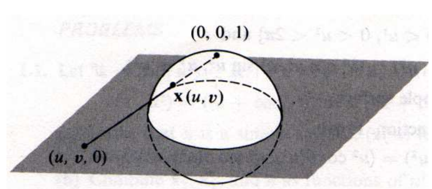
where a, b are positive constants. Show that $\mathbf{x}$ is a regular surface by computing $|\mathbf{x_u} \times \mathbf{x_v}|$. (Notation: $\mathbf{x_u} = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}$, etc.)



2. (a) Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined as follows: For each $(u, v) \in \mathbb{R}^2$, let $\mathbf{x}(u, v)$ be the point of intersection of the line through $(0, 0, 1)$ and $(u, v, 0)$ with the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. (See the figure below.) Show that

$$\mathbf{x}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

Hint: By parameterizing the line through $(0, 0, 1)$ and $(u, v, 0)$, show that $\mathbf{x}(u, v)$ is of the form, $\mathbf{x}(u, v) = (0, 0, 1) + t(u, v, -1)$ for some t, and then determine t. (Remark: The map $\mathbf{x}$ is a proper patch which covers all of S^2 except the north pole. The inverse map $\mathbf{x}^{-1}$ is called the *stereographic projection* of S^2 onto $\mathbb{R}^2$.)



(b) Let $\mathbf{y}(u,v) = (u, v, \sqrt{1-u^2-v^2})$ be the Monge patch associated with the upper hemisphere $z = \sqrt{1-x^2-y^2}$. Write out explicitly the overlap map $\mathbf{y}^{-1} \circ \mathbf{x}$, where $\mathbf{x}$ is as in part (a), and observe that it is smooth.

3. Carefully apply the "inverse image theorem" to show that the graph of the equation: $xy + yz + zx = 1$ is a smooth surface.

4. Let $\bar{x} : U \to M$ and $\bar{y} : V \to M$ be overlapping patches in M containing the point p. Consider the tangent vector $X \in T_pM$ with respect to both coordinate vector bases, $X = \sum_i X^i \bar{x}_i = \sum_j \tilde{X}^j \bar{y}_j$. Show that the components of X in the two coordinate ssytems are related by,

$$\tilde{X}^k = \sum_{i=1}^{2} \frac{\partial v^k}{\partial u^i} X^i,$$

where $\left[\frac{\partial v^k}{\partial u^i}\right]$ is the Jacobian matrix of the change of coordinate map $f = \bar{y}^{-1} \circ x$, $f(u^1, u^2) = (v^1(u^1, u^2), v^2(u^1, u^2))$.

5. Let M and N be surfaces. A map $f : M \to N$ is said to be smooth provided for each $p \in M$ there is a proper patch $\bar{x} : U \to M$ containing p and a proper patch $\bar{y} : V \to N$ containing $f(p)$, with $f(\bar{x}(U)) \subset \bar{y}(V)$ such that $\hat{f} = \bar{y}^{-1} \circ f \circ \bar{x} : U \to V$ is a smooth Euclidean map.

   Let $f : M \to N$ be smooth. For a fixed point $p \in M$ we define a mapping between tangent spaces $df : T_pM \to T_{f(p)}N$, called the differential of f, as follows. For any $X \in T_pM$, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve in M such that $\alpha(0) = p$ and $\alpha'(0) = X$. Then $\beta = f \circ \alpha : (-\epsilon, \epsilon) \to N$ will be a smooth curve in N passing through $f(p)$. We now define: $df(X) = \beta'(0)$.

   Problem: Show that df is a linear map, and, in fact, that the matrix representing df with respect to the coordinate bases $\{\bar{x}_i\}$ at p and $\{\bar{y}_j\}$ at $f(p)$ is the Jacobian matrix of $\hat{f} = \bar{y}^{-1} \circ f \circ \bar{x}$ (evaluated at $\bar{x}^{-1}(p)$). (Hint: Show that $\hat{\beta}(t) = \hat{f}(u^1(t), u^2(t))$, where $\hat{\beta}(t) = \bar{y}^{-1} \circ \beta(t) = (v^1(t), v^2(t))$ and $\alpha(t) = \bar{x}(u^1(t), u^2(t))$. Then apply the chain rule.)

Chapter 4

The First Fundamental Form (Induced Metric)

We begin with some definitions from linear algebra.

Definition. Let V be a vector space (over $I\!R$). A *bilinear form* on V is a map of the form $B : V \times V \to I\!R$ which is *bilinear*, i.e. linear in each "slot",

$$\begin{aligned} B(aX + bY, Z) &= aB(X, Z) + bB(Y, Z), \\ B(X, cY + dZ) &= cB(X, Y) + dB(X, Z). \end{aligned}$$

A bilinear form B is *symmetric* provided $B(X, Y) = B(Y, X)$ for all $X, Y \in V$.

Definition. Let V be a vector space. An *inner product* on V is a symmetric bilinear form $\langle\ ,\ \rangle : V \times V \to I\!R$ which, in addition, is positive definite,

$$\langle X, X\rangle \geq 0 \text{ for all } X \text{ and } = 0 \text{ iff } X = 0\,.$$

Example. $\langle\ ,\ \rangle : T_pI\!R^3 \times T_pI\!R^3 \to I\!R$,

$$\langle X, Y\rangle = X \cdot Y \quad \text{(usual Euclidean dot product)}\,.$$

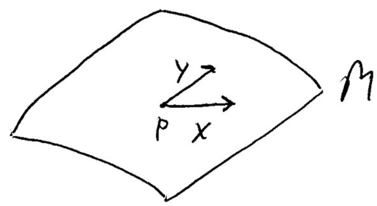
EXERCISE 4.1. Verify carefully that the Euclidean dot product is indeed an inner product.

Definition. Let M be a surface. A *metric* on M is an assignment, to each point $p \in M$, of an inner product $\langle\ ,\ \rangle : T_pM \times T_pM \to I\!R$.

Because our surfaces sit in Euclidean space, they inherit in a natural way, a metric called the *induced metric* or *first fundamental form.*

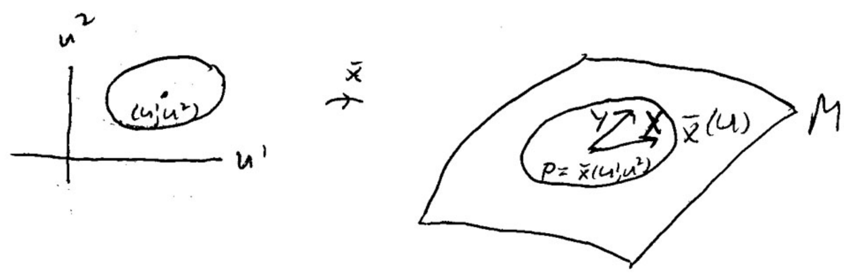
Definition. Let M be a surface. The *induced metric* (or *first fundamental form*) of M is the assignment to each $p \in M$ of the inner product,

$$\begin{aligned} \langle\ ,\ \rangle : T_pM &\times T_pM \to I\!R\,, \\ \langle X, Y\rangle &= X \cdot Y \quad \text{(ordinary scalar product of } X \text{ and } Y \\ &\qquad\qquad \text{viewed as vectors in } I\!R^3 \text{ at } p) \end{aligned}$$



I.e., the induced metric is just the Euclidean dot product, restricted to the tangent spaces of M. We will only consider surfaces in the induced metric. Just as the Euclidean dot product contains all geometric information about $I\!R^3$, the induced metric contains all geometric information about M, as we shall see.

<u>The Metric in a Coordinate Patch</u>.



Let $\mathbf{x} : U \subset I\!R^2 \to M \subset I\!R^3$ be a proper patch in M. Let $p \in \mathbf{x}(U)$ be any point in $\mathbf{x}(U)$, $p = \mathbf{x}(u^1, u^2)$, and let $X, Y \in T_pM$. Then,

$$\begin{aligned} X &= X^1 \frac{\partial \mathbf{x}}{\partial u^1} + X^2 \frac{\partial \mathbf{x}}{\partial u^2} = X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2\,, \\ X &= \sum_i X^i \mathbf{x}_i\,, \quad \mathbf{x}_i = \mathbf{x}_i(u^1, u^2)\,, \end{aligned}$$

and similarly,

$$Y = \sum_j Y^j \mathbf{x}_j \,.$$

Then,

$$\begin{aligned} \langle X, Y \rangle &= \langle \sum_i X^i \mathbf{x}_i, \sum_j Y^j \mathbf{x}_j \rangle \\ &= \sum_{i,j} X^i Y^j \langle \mathbf{x}_i, \mathbf{x}_j \rangle. \end{aligned}$$

The *metric components* are the functions $g_{ij} : U \to I\!R$, $1 \leq i, j \leq 2$, defined by

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle \,, \quad g_{ij} = g_{ij}(u^1, u^2) \,. \tag{4.1}$$

Thus, in coordinates,

$$\langle X, Y \rangle = \sum_{i,j=1}^{2} g_{ij} X^i Y^j \,. \tag{4.2}$$

Note that the metric in $\mathbf{x}(U)$ is completely determined by the g_{ij}'s. The metric components may be displayed as a 2×2 matrix,

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \,.$$

<u>Note</u>: $g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{x}_j, \mathbf{x}_i \rangle = g_{ji}$. Hence, the matrix of metric components is symmetric; and there are only three distinct components,

$$g_{11} = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle, \;\; g_{12} = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = g_{21}, \;\; g_{22} = \langle \mathbf{x}_2, \mathbf{x}_2 \rangle \,.$$

<u>Notations</u>:

1. Gauss: $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$.

2. $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. Then one writes:

$$g_{uu} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \; g_{uv} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \;\; g_{vv} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle \,.$$

Example. Consider the parameterization of S_r^2 in terms of geographic coordinates,

$$\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r\sin\theta\sin\phi, r\cos\theta) \ ,$$

$0 < \theta < \pi, \quad 0 < \phi < 2\pi$. We compute the metric components in these coordinates. We have,

$$\begin{aligned}
\mathbf{x}_\theta \quad &= \frac{\partial \mathbf{x}}{\partial \theta} = r(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \\
\mathbf{x}_\phi \quad &= r(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) \, , \\
g_{\theta\theta} \quad &= \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle \\
&= r^2[\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta] \\
&= r^2(\cos^2\theta + \sin^2\theta) = r^2 \, , \\
g_{\theta\phi} \quad &= \langle \mathbf{x}_\theta, \mathbf{x}_\phi \rangle \\
&= r^2[-\cos\theta\cos\phi\sin\theta\sin\phi + \cos\theta\sin\phi\sin\theta\cos\phi] \\
&= 0 \qquad \text{(geometric significance?)} \, , \\
g_{\phi\phi} \quad &= r^2[\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi] \\
&= r^2\sin^2\theta \, .
\end{aligned}$$

Thus,

$$[g_{ij}] = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{bmatrix}$$

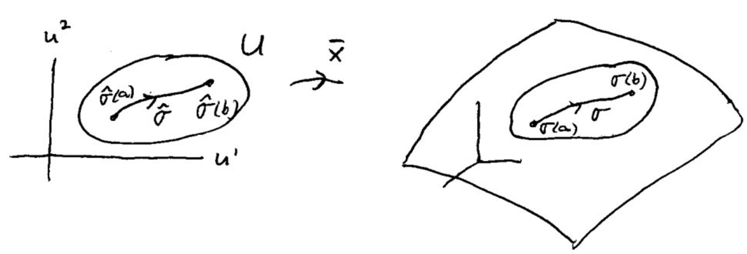
Length and Angle Measurement in M.

Let $\sigma : [a,b] \to M \subset \mathbb{R}^3$ be a smooth curve in a surface M. Viewed as a curve in $\mathbb{R}^3$, $\quad \sigma(t) = (x(t), y(t), z(t))$, we can compute its length by the formula,

$$\begin{aligned}
\text{Length of } \sigma &= \int_a^b \left|\frac{d\sigma}{dt}\right| dt \\
&= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt
\end{aligned}$$

But this formula does not make sense to creatures living in the surface: x, y, z are Euclidean *space* coordinates. Creatures living in the surface must use *surface* coordinates – i.e., we must express σ in terms of surface coordinates.

Let $\mathbf{x} : U \to M \subset I\!R^3$ be a proper patch in M and suppose σ is contained in this patch, $\sigma \subset \mathbf{x}(U)$:



We express σ in terms of coordinates: $\hat{\sigma} = x^{-1} \circ \sigma : [a, b] \to U \subset I\!R^2$, $\hat{\sigma}(t) = (u'(t), u^2(t))$, i.e,

$$\hat{\sigma} : \begin{array}{l} u^1 = u^1(t) \\ u^2 = u^2(t) \end{array} , \quad a \le t \le b.$$

Then, $\sigma = \mathbf{x} \circ \hat{\sigma}$, i.e., $\sigma(t) = \mathbf{x}(\hat{\sigma}(t))$, hence (see Equation (3.2)),

$$\sigma(t) = \mathbf{x}(u^1(t), u^2(t)) .$$

By the chain rule,

$$\begin{aligned}
\frac{d\sigma}{dt} &= \frac{\partial \mathbf{x}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{du^2}{dt} \\
&= \frac{du^1}{dt}\mathbf{x}_1 + \frac{du^2}{dt}\mathbf{x}_2 ,
\end{aligned}$$

or,

$$\frac{d\sigma}{dt} = \sum_i \frac{du^i}{dt}\mathbf{x}_i\,.$$

This shows that $\dfrac{du^i}{dt}$, $i = 1, 2$, are the components of the velocity vector with respect to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$.

Computing the dot product,

$$\begin{aligned}
\langle \frac{d\sigma}{dt}, \frac{d\sigma}{dt} \rangle &= \langle \sum_i \frac{du^i}{dt}\mathbf{x}_i, \sum_j \frac{du^j}{dt}\mathbf{x}_j \rangle \\
&= \sum_{i,j} \frac{du^i}{dt}\frac{du^j}{dt}\langle \mathbf{x}_i, \mathbf{x}_j \rangle \\
&= \sum_{i,j=1}^{2} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}.
\end{aligned}$$

Hence, for the *speed* in surface coordinates, we have,

$$\left|\frac{d\sigma}{dt}\right| = \sqrt{\sum_{i,j=1}^{2} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\,.$$

For length, we then have,

$$\begin{aligned}
\text{Length of } \sigma &= \int_a^b \sqrt{\sum_{i,j} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\; dt \\
&= \int_a^b \sqrt{g_{11}\left(\frac{du^1}{dt}\right)^2 + 2g_{12}\frac{du^1}{dt}\frac{du^2}{dt} + g_{22}\left(\frac{du^2}{dt}\right)^2}\; dt \qquad (4.3)
\end{aligned}$$

Arc length element. Let s denote arc length along σ; s can be computed in terms of t as follows. $s = s(t),\ a \le t \le b$,

$$\begin{aligned} s(t) &= \text{length of } \sigma \text{ from time } a \text{ to time } t \\ &= \int_a^t \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}}\, dt\,. \end{aligned}$$

and hence,

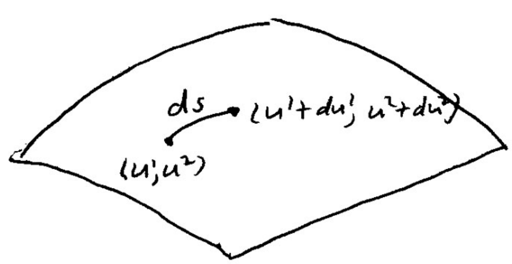
$$\frac{ds}{dt} = \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}}\,.$$

In terms of differentials,

$$\begin{aligned} ds &= \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}} dt \\ ds^2 &= \left(\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}\right) dt^2 \\ &= \sum_{i,j} g_{ij} \left(\frac{du^i}{dt} dt\right)\left(\frac{du^j}{dt} dt\right) \end{aligned}$$

and we arrive at the expression for the *arc length element* in terms of the metric components,

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} du^i du^j \tag{4.4}$$



Heuristically, moving from a point with coordinates (u_1, u_2) to the near by point $(u_1 + du_1, u_2 + du_2)$, produces an arc length ds given by (4.4).

Traditionally, one displays the metric (or, metric components g_{ij}) by writing out the arc length element.

Notations:

$$
\begin{aligned}
ds^2 = &\ g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 \\
ds^2 = &\ g_{uu}du^2 + 2g_{uv}dudv + g_{vv}dv^2 \\
&\ (u^1 = u, u^2 = v) \\
ds^2 = &\ Edu^2 + 2Fdudv + Gdv^2 \qquad \text{(Gauss)}.
\end{aligned}
$$

Remark. These expressions for arc length element of a surface M generalize the expression for the arc length element in the Euclidean u-v plane we encounter in calculus,

$$ds^2 = du^2 + dv^2$$

(i.e. $g_{uu} = 1, \;\; g_{uv} = 0, \;\; g_{vv} = 1$).

Example. Write out the arc length element for the sphere S^2_r parameterized in terms of geographic coordinates,

$$\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \, .$$

We previously computed the g_{ij}'s,

$$[g_{ij}] = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix},$$

i.e., $\;\; g_{\theta\theta} = r^2, \;\; g_{\theta\phi} = g_{\phi\theta} = 0, \;\; g_{\phi\phi} = r^2 \sin^2\theta$. So,

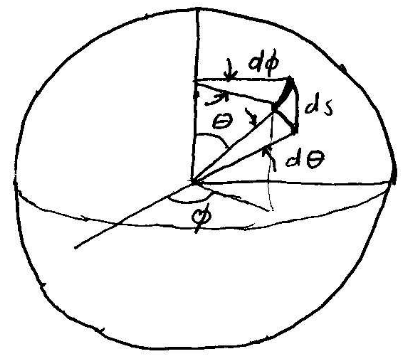
$$
\begin{aligned}
ds^2 &= g_{\theta\theta}d\theta^2 + 2g_{\theta\phi}d\theta d\phi + g_{\phi\phi}d\phi^2 \\
ds^2 &= r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, .
\end{aligned}
$$

But this expression is familiar from calculus as the arc length element which can be derived from heuristic geometric considerations:

$$ds^2 = d\ell_1^2 + d\ell_2^2.$$

$$d\ell_1 = r d\theta, \quad d\ell_2 = r \sin\theta d\phi$$

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

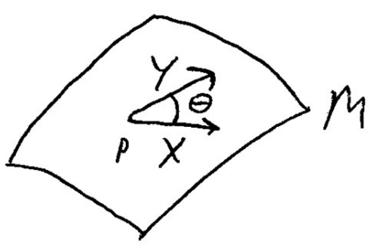


EXERCISE 4.2. Consider the parameterization of the x-y plane in terms of *polar coordinates*,

$$\mathbf{x} : \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = 0 \end{array} \quad , \; 0 < r < \infty, \;\; 0 < \theta < 2\pi \; ,$$

i.e., $\mathbf{x}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, $0 < r < \infty$, $0 < \theta < 2\pi$. Compute the g_{ij}'s with respect to these coordinates. Show that the arc length element in this case is: $ds^2 = dr^2 + r^2 d\theta^2$.

Angle Measurement.



$$X = \sum_i X^i \mathbf{x}_i,$$

$$Y = \sum_j Y^j \mathbf{x}_j,$$

$$\begin{aligned} \cos\theta &= \frac{\langle X, Y\rangle}{|X||Y|} \\ &= \frac{\sum g_{ij} X^i Y^j}{\sqrt{\sum g_{ij} X^i X^j}\,\sqrt{\sum g_{ij} Y^i Y^j}} \; . \end{aligned}$$

Example. Determine the angle between the coordinate vectors $\mathbf{x}_1 = \dfrac{\partial \mathbf{x}}{\partial u^1}$ and $\mathbf{x}_2 = \dfrac{\partial \mathbf{x}}{\partial u^2}$ in terms of the g_{ij}'s.

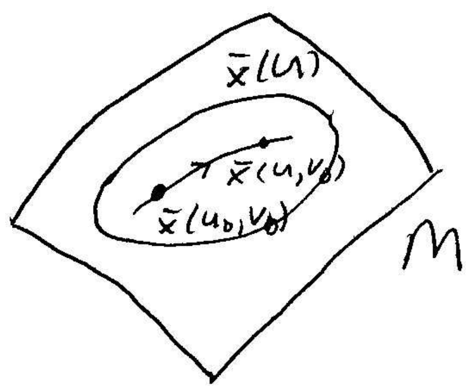
$$\cos\theta = \frac{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle}{|\mathbf{x}_1|\ \mathbf{x}_2|} = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

($|\mathbf{x}_1| = \sqrt{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} = \sqrt{g_{11}}$, etc.)

<u>The Metric is intrinsic:</u>

This discussion is somewhat heuristic. We claim that the g_{ij}'s are *intrinsic*, i.e. in principle they can be determined by measurements made in the surface.

Let $\mathbf{x} : U \to M$ be a proper patch in M; $\mathbf{x} = \mathbf{x}(u^1, u^2) = \mathbf{x}(u, v)$ (i.e., $u^1 = u$, $u^2 = v$. Consider the coordinate curve $u \xrightarrow{\sigma} \mathbf{x}(u, v_0)$ passing through $\mathbf{x}(u_0, v_0)$.



Let $s = s(u)$ be the arc length function along σ, i.e.,

$$\begin{aligned} s(u) &= \text{length of } \sigma \text{ from } u_0 \text{ to } u \\ &= \int_{u_0}^{u} \left| \frac{\partial \mathbf{x}}{\partial u} \right| du \\ &= \int_{u_0}^{u} \sqrt{g_{uu}} du \qquad \left(\left| \frac{\partial \mathbf{x}}{\partial u} \right| = \sqrt{g_{uu}} \right). \end{aligned}$$

By making length measurements in the surface the function $s = s(u)$ can be determined known. Then by calculus, the derivative,

$$\frac{ds}{du} = \sqrt{g_{uu}}$$

is known. Therefore $g_{11} = g_{uu}$, and similarly $g_{22} = g_{vv}$, can in principal be determined by measurements made in the surface.

The metric component g_{12} can then be determined by angle measurement,

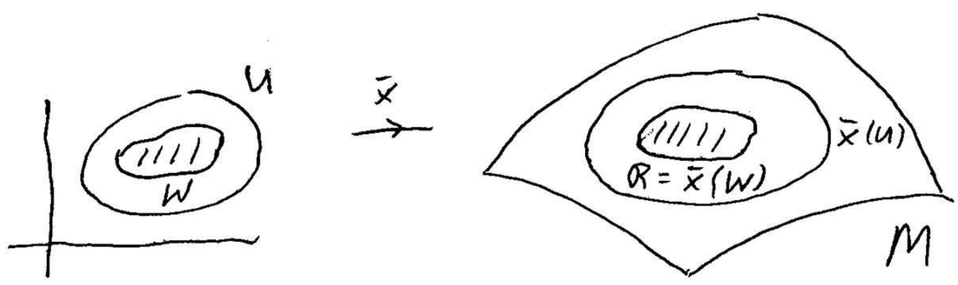
$$\begin{aligned} g_{12} &= \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = |\mathbf{x}_1||\mathbf{x}_2| \cos\theta \\ &= \sqrt{g_{11}}\sqrt{g_{22}} \cdot \cos(\text{angle between } \mathbf{x}_1, \mathbf{x}_2) \ . \end{aligned}$$

Hence g_{12} is also measurable. Thus *all* metric components can be determined by measurements made in the surface, i.e.

the metric components and all quantities determined from them are intrinsic.

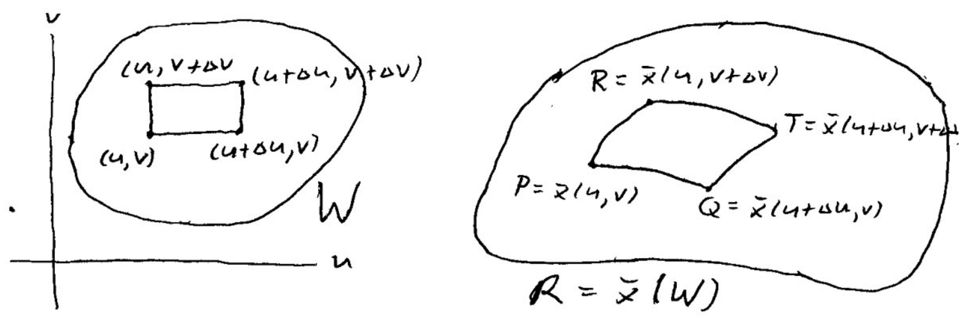
Surface Area.

Let M be a surface, and let $\mathbf{x} : U \to M$ be a proper patch in M. Consider a bounded region $\mathcal{R}$ contained in $\mathbf{x}(U)$; we have $\mathcal{R} = \mathbf{x}(W)$ for some bounded region W in U:



We want to obtain (i.e. heuristically motivate) a formula for the area of $\mathcal{R} = \mathbf{x}(W)$.

Restrict attention to $\mathcal{R} = \mathbf{x}(W)$; partition W into small rectangles:



Let ΔS = area of the small patch corresponding to the coordinate rectangle. Then,

$$\begin{aligned} \Delta S &\approx \text{area of the parallelogram spanned by } \vec{PQ} \text{ and } \vec{PR}\,, \\ \Delta S &\approx |\,\vec{PQ} \times \vec{PR}\,|\,. \end{aligned}$$

But,

$$\begin{aligned} \vec{PQ} &= \mathbf{x}(u+\Delta u, v) - \mathbf{x}(u,v) \approx \frac{\partial \mathbf{x}}{\partial u}\Delta u\,, \\ \vec{PR} &= \mathbf{x}(u, v+\Delta v) - \mathbf{x}(u,v) \approx \frac{\partial \mathbf{x}}{\partial v}\Delta v\,, \end{aligned}$$

and thus,

$$\begin{aligned} \Delta S &\approx \left|\frac{\partial \mathbf{x}}{\partial u}\Delta u \times \frac{\partial \mathbf{x}}{\partial v}\Delta v\right| \\ &\approx \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| \Delta u \Delta v\,. \end{aligned}$$

The smaller the increments Δu and Δv, the better the approximation.

$$\begin{aligned} dS &= \text{the area element of the surface corresponding to the} \\ &\quad\ \text{coordinate increments } du,\ dv\,, \\ dS &= \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| du\ dv\,. \end{aligned}$$

To obtain the total area of $\mathcal{R}$, we must sum up all these area elements - but the summing up process is integration:

$$\begin{aligned} \text{Area of } \mathcal{R} &= \iint dS\,, \\ \text{Area of } \mathcal{R} &= \iint_W \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du\, dv\,, \end{aligned}$$

where $\mathcal{R} = \mathbf{x}(W)$.

This is a perfectly reasonable formula for computing surface area - but *not* for 2-dimensional creatures living in the surface. It involves the cross product which is an $I\!R^3$ concept. We now show how this area formula can be expressed in an *intrinsic* way (i.e. involving the g_{ij}'s).

Using generic notation, $u^1 = u,\ u^2 = v,\ \mathbf{x} = \mathbf{x}(u^1, u^2)$ we write,

$$\begin{aligned} \text{Area of } \mathcal{R} &= \iint_W \left| \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \right| du^1 du^2 \\ &= \iint_W |\mathbf{x}_1 \times \mathbf{x}_2| du^1 du^2 \end{aligned}$$

Now introduce the notation,

$$g := \det[g_{ij}], \quad g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle\,.$$

Claim: $g = |\mathbf{x}_1 \times \mathbf{x}_2|^2$

Proof. Recall the vector identity,

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2\,.$$

Hence,

$$\begin{aligned} |\mathbf{x}_1 \times \mathbf{x}_2|^2 &= |\mathbf{x}_1|^2 |\mathbf{x}_2|^2 - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2 \\ &= \langle \mathbf{x}_1, \mathbf{x}_1 \rangle \langle \mathbf{x}_2, \mathbf{x}_2 \rangle - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2 \\ &= g_{11} g_{22} - g_{12}^2 = g, \end{aligned}$$

But,

$$g = \det[g_{ij}] \;\; = \det \begin{bmatrix} g_{11} & g_{12} \\ \\ g_{21} & g_{22} \end{bmatrix} = g_{11}g_{22} - g_{12}^2\,,$$

where we have used $g_{21} = g_{12}$. □

Thus, the surface area formula may be expressed as,

$$\text{Area of } \mathcal{R} = \iint_W \sqrt{g}\, du^1 du^2 \qquad (\mathcal{R} = \mathbf{x}(W)) \tag{4.5}$$

$$= \iint_W dS \tag{4.6}$$

where,

$$dS = \sqrt{g}\, du^1 du^2\,. \tag{4.7}$$

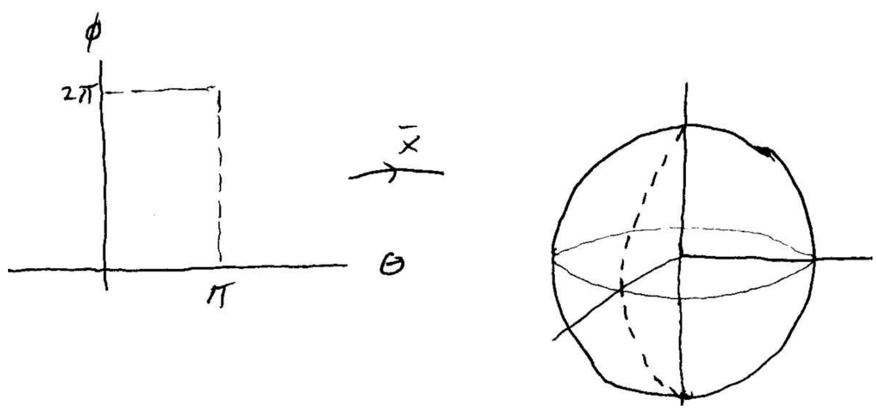
Example. Compute the area of the sphere of radius r.

$$S_r^2 : x^2 + y^2 + z^2 = r^2\,.$$

Parameterize with respect to geographical coordinates, $\mathbf{x} : U \to S_r^2$,

$$\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta)\,,$$

$U : 0 < \theta < \pi,\ 0 < \phi < 2\pi$.



We have,

$$\text{Area of } S_r^2 = \iint_U dS\,, \quad \text{where} \quad dS = \sqrt{g}d\theta d\phi\,.$$

Now,

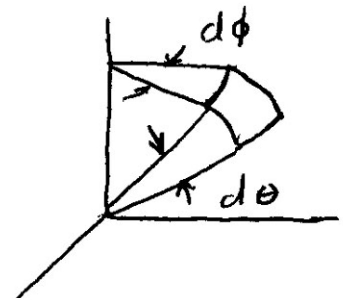
$$\begin{aligned} g &= \det[g_{ij}] = \det \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} \\ &= \det \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix} \\ g &= r^4 \sin^2\theta \end{aligned}$$

Thus,

$$dS = \sqrt{r^4 \sin^2\theta}\, d\theta d\phi = r^2 \sin\theta d\theta d\phi$$

Side remark: This expression for the surface area element of a sphere is familiar from calculus or physics where it is usually derived by heuristic considerations:

$$\begin{aligned} dS &= d\ell_1 d\ell_2, \\ d\ell_1 &= r d\theta,\ d\ell_2 = r\sin\theta d\phi \\ dS &= (r d\theta)(r \sin\theta d\phi) \\ &= r^2 \sin\theta d\theta d\phi. \end{aligned}$$



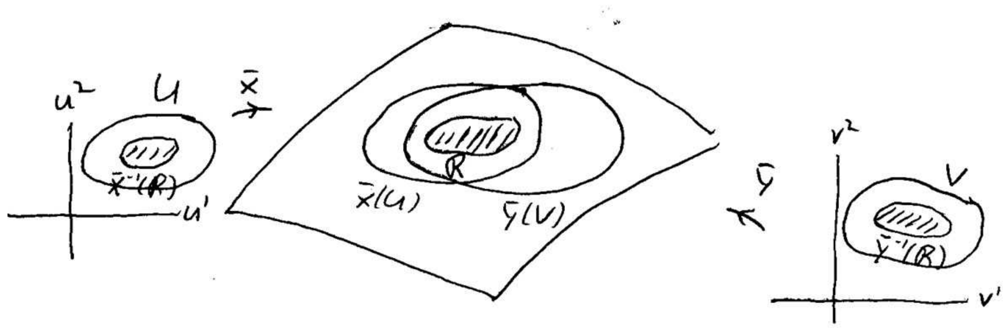
Continuing the computation of the surface area of S_r^2,

$$\begin{aligned} \text{Area of } S_r^2 &= \iint_U r^2 \sin\theta d\theta d\phi = \iint_{\overline{U}} r^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta d\theta d\phi = \int_0^{2\pi} r^2[-\cos\theta]|_0^{\pi}\ d\phi \\ &= \int_0^{2\pi} 2r^2 d\phi = 2r^2\phi|_0^{2\pi} = 4\pi r^2\,. \end{aligned}$$

The surface area formula involves a choice of coordinates, i.e. a choice of proper patch. It is important to recognize that the formula is independent of this choice.

Proposition 4.1. *The area formula is independent of the choice of coordinate patch.*

Let $\mathbf{x} : U \to M$, $\mathbf{y} : V \to M$ be proper patches, and suppose $\mathcal{R}$ is contained in $\mathbf{x}(U) \cap \mathbf{y}(V)$:



Set,

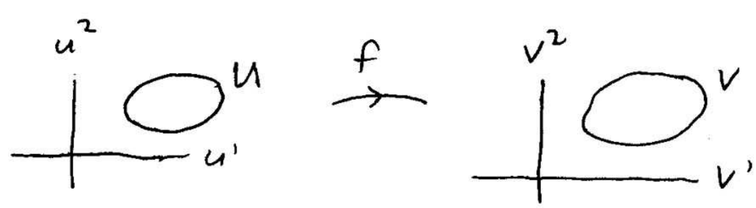
$$\begin{aligned} g_{ij} &= \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \quad g = \det[g_{ij}]\,, \\ \tilde{g}_{ij} &= \langle \mathbf{y}_i, \mathbf{y}_j \rangle, \quad \tilde{g} = \det[\tilde{g}_{ij}]\,. \end{aligned}$$

Then the claim is that,

$$\iint_{\mathbf{x}^{-1}(\mathcal{R})} \sqrt{g}\,du^1du^2 = \iint_{\mathbf{y}^{-1}(\mathcal{R})} \sqrt{\tilde{g}}\,dv^1dv^2$$

Proof. The proof is an application of the change of variable formula for double integrals.

Let $f : U \subset I\!R^2 \to V \subset I\!R^2$ be a diffeomorphism, where U, V are bounded regions in $I\!R^2$.



$$f : \begin{array}{l} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{array} .$$

Then, the change of variable formula for double integrals is as follows,

$$\begin{aligned} \iint_V h(v^1, v^2) dv^1 dv^2 &= \iint_U h \circ f(u^1, u^2) |\det Df| du^1 du^2 \\ &= \iint_U h(f^1(u^1, u^2), f^1(u^1, u^2)) \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| du^1 du^2 \end{aligned} ,$$

or, in briefer notation,

$$\iint_V h dv^1 dv^2 = \iint_U h \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| du^1 du^2 .$$

In the case at hand, $f = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(\mathcal{R}) \to \mathbf{y}^{-1}(\mathcal{R})$, and $h = \sqrt{\tilde{g}}$. So, by the change of variable formula,

$$\iint_{y^{-1}(\mathcal{R})} \sqrt{\tilde{g}} dv^1 dv^2 = \iint_{\mathbf{x}^{-1}(\mathcal{R})} \sqrt{\tilde{g}} \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| du^1 du^2$$

Thus, to complete the proof, it suffices to establish the following lemma.

Lemma 4.2. $g = \det[g_{ij}]$, $\tilde{g} = \det[\tilde{g}_{ij}]$. *are related by,*

$$\sqrt{g} = \sqrt{\tilde{g}} \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| .$$

Proof of the lemma: It follows from Exercise 3.4 that,

$$\frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} = \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \frac{\partial \mathbf{y}}{\partial v^1} \times \frac{\partial \mathbf{y}}{dv^2}$$

or,

$$\mathbf{x}_1 \times \mathbf{x}_2 = \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \mathbf{y}_1 \times \mathbf{y}_2 .$$

Hence,

$$\begin{aligned} g &= \det[g_{ij}] = |\mathbf{x}_1 \times \mathbf{x}_2|^2 \\ &= \left(\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right)^2 |\mathbf{y}_1 \times \mathbf{y}_2|^2 \\ &= \left(\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right)^2 \tilde{g}\,. \end{aligned}$$

Taking square roots yields the result. □

EXERCISE 4.3. Consider the torus of large radius R and small radius r described in Exercise 3.3. Use the *intrinsic* surface area formula and the parameterization given in Exercise 3.3 to compute the surface area of the torus. Answer: $4\pi^2 Rr$.

EXERCISE 4.4. Let $f : U \subset I\!R^2 \to I\!R$ be a smooth function of two variables. Let M be the graph of $f|_w = \{(x, y, z) \in I\!R^3 : z = f(x, y), (x, y) \in W\}$, where W is a bounded subset of U. Derive the following standard formula from calculus for the surface area of M,

$$\text{Area of } M = \iint_W \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy\,,$$

by considering the Monge patch associated to f.

<u>More Tensor Analysis</u>

Let $\mathbf{x} : U \to M$, $\mathbf{y} : V \to M$ be overlapping patches in a surface M, $W := \mathbf{x}(U) \cap \mathbf{y}(V) \neq \emptyset$. Let $f = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$,

$$f : \begin{array}{l} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{array}$$

be the smooth overlap map, cf., Proposition 3.4. Introduce the metric components with respect to each patch,

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \quad \tilde{g}_{ij} = \langle \mathbf{y}_i, \mathbf{y}_j \rangle\,.$$

How are these metric components related on the overlap?

EXERCISE 4.5. Show that,

$$g_{ij} = \sum_{a,b=1}^{2} \tilde{g}_{ab} \frac{\partial v^a}{\partial u^i} \frac{\partial v^b}{\partial u^j}, \quad i, j = 1, 2.$$

These equations can be expressed as a single matrix equation,

$$[g_{ij}] = \left[\frac{\partial v^a}{\partial u^i}\right]^t [\tilde{g}_{ab}] \left[\frac{\partial v^b}{\partial u^j}\right].$$

Taking determinants we obtain,

$$\begin{aligned}
g &= \det[g_{ij}] = \det[*]^t[*][*] \\
&= \det[*]^t \det[*] \det[*] \\
&= \det\left[\frac{\partial v^a}{\partial u^i}\right] \det[\tilde{g}_{ij}] \det\left[\frac{\partial v^b}{\partial u^j}\right] \\
&= \tilde{g}(\det Df)^2 \\
g &= \tilde{g}\left[\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right]^2,
\end{aligned}$$

our second derivation of this formula.

Remark. Interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$ above we obtain,

$$\tilde{g}_{ab} = \sum_{i,j} g_{ij} \frac{\partial u^i}{\partial v^a} \frac{\partial u^j}{\partial v^b},$$

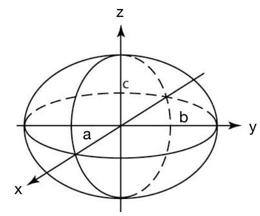
which involves the Jacobian of f^{-1}. Compare this "transformation law" for the metric components to the transformation law for vector components considered in Exercise 3.8. Vector fields are "contravariant" tensors. The metric $\langle\ ,\ \rangle$ is a "covariant" tensor.

Additional Chapter 4 Exercises

1. Compute the metric components $g_{11} = g_{uu}$, $g_{12} = g_{uv}$, $g_{22} = g_{vv}$ for the helicoid (cf. Exercise #1 in Chapter 3 Additional Exercises).

2. Consider the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,, \quad (a, b, c > 0)$$



   (a) Introduce 'geographic' coordinates θ, ϕ on the ellipsoid by slightly modifying the equations for spherical coordinates.

   (b) Write down the proper patch $\mathbf{x}$ associated with the geographic coordinates above, and compute the g_{ij}'s ($g_{\theta\theta}$, etc.).

3. Let $U = \{(r, \theta) \in \mathbb{R}^2 : 0 < r < \infty, 0 < \theta < 2\pi\}$. The map $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{x}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$ is a proper patch which parameterizes the x-y plane in terms of polar coordinates.

   (a) Compute the metric components g_{ij} (i.e., g_{rr}, etc.) with respect to this parameterization.

   (b) The polar equation: $r = r(\theta)$, $\alpha \le \theta \le \beta$, describes a curve σ in the x-y plane. Use the length formula, Equation (4.2) on p. 72, to show,

$$\text{Length of } \sigma = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta\,,$$

   by parmeterizing σ appropriately.

   (c) Use part (b) to compute the length of the *logarithmic spiral* $r = e^{-\theta}$,
   $0 \le \theta < \infty$.

4. Consider the map $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $U = \{(0 < t < 2\pi, -\pi < \theta < \pi\}$,

$$\mathbf{x}(t, \theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t)$$

   which parameterizes the torus, as described in Exercise 3.3, on p. 53.

(a) Compute the metric components g_{ij} (i.e., g_{tt}, etc.) with respect to this parameterization.

(b) Use the g_{ij}'s to compute the surface area element dS and find the surface area of the torus.

5. *Surfaces of revolution.* Rotate the regular curve $\sigma : x = r(t), z = z(t)$, $a \leq t \leq b$, in the x-z plane about the z-axis to obtain a surface of revolution M (cf., Chapter 3, p 52f).

   (a) Compute the metric components $g_{11} = g_{rr}$, etc.

   (b) Use part (a) and Equation 4.5 to show,

$$\text{Surface area of } M = 2\pi \int_a^b r\sqrt{(r')^2 + (z')^2}\, dt.$$

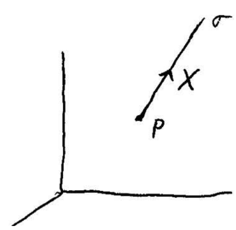
   (c) *Pappus' Theorem.* Let s denote arc length along σ, $\ell =$ the length of σ, and $\rho(s) = r(t(s))$ be the radial function reparameterized in terms of arc length. Use part (b) to show that the surface area of $M = 2\pi\bar{\rho}\ell$, where $\bar{\rho}$ is the *average value* of $\rho = \rho(s)$, $\bar{\rho} = \int_0^\ell \rho(s)\ ds/\ell$. (Remark: Note that, by symmetry considerations, $\bar{\rho} = R$ for the torus in part (c). This provides a check on part (c)).

Chapter 5

The Second Fundamental Form

Directional Derivatives in $\mathbb{R}^3$.

Let $f : U \subset I\!R^3 \to I\!R$ be a smooth function defined on an open subset of $I\!R^3$. Fix $p \in U$ and $X \in T_p I\!R^3$. The *directional derivative* of f at p in the direction X, denoted $D_X f$ is defined as follows. Let $\sigma : I\!R \to I\!R^3$ be the parameterized straight line, $\sigma(t) = p + tX$ (note $\sigma(0) = p$ and $\sigma'(0) = X$):



Then,

$$
\begin{aligned}
D_X f &= \frac{d}{dt} f \circ \sigma(t)|_{t=0} \\
&= \frac{d}{dt} f(p + tX)|_{t=0} \\
&\left(= \lim_{t \to 0} \frac{f(p + tX) - f(p)}{t} \right).
\end{aligned}
$$

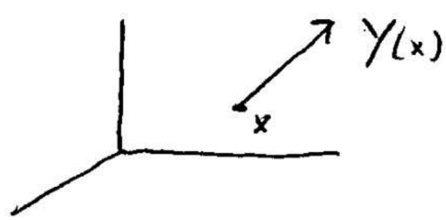
Proposition 5.1. *The directional derivative is given by the following formula,*

$$\begin{aligned} D_X f &= X \cdot \nabla f(p) \\ &= (X^1, X^2, X^3) \cdot \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right) \\ &= X^1 \frac{\partial f}{\partial x^1}(p) + X^2 \frac{\partial f}{\partial x^2}(p) + X^3 \frac{\partial f}{\partial x^3}(p) \\ &= \sum_{i=1}^{3} X^i \frac{\partial f}{\partial x^i}(p). \end{aligned}$$

Proof. Chain rule!

Vector Fields on $I\!R^3$. A vector field on $I\!R^3$ is a rule which assigns to each point of $I\!R^3$ a vector at the point,

$$x \in I\!R^3 \to Y(x) \in T_x I\!R^3$$



Analytically, a vector field is described by a mapping of the form,

$$\begin{aligned} Y : U &\subset I\!R^3 \to I\!R^3, \\ Y(x) &= (Y^1(x), Y^2(x), Y^3(x)) \in T_x I\!R^3 . \end{aligned}$$

The components of Y are the real valued functions: $Y^i : U \to I\!R$, $i = 1, 2, 3$.

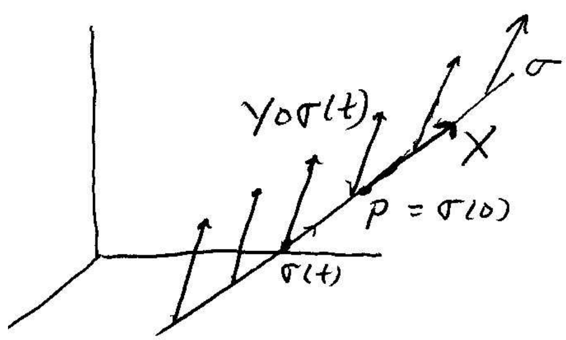
Example. $Y : I\!R^3 \to I\!R^3$, $Y(x, y, z) = (y+z, z+x, x+y)$. E.g., $Y(1, 2, 3) = (5, 4, 3)$, etc. $Y^1 = y + z$, $Y^2 = z + x$, and $Y^3 = x + y$.

The *directional derivative of a vector field* is defined in a manner similar to the directional derivative of a function: Fix $p \in U, \quad X \in T_p\mathbb{R}^3$. Let $\sigma : \mathbb{R} \to \mathbb{R}^3$ be the parameterized line $\sigma(t) = p + tX \ \ (\sigma(0) = p, \sigma'(0) = X)$. Then $t \to Y \circ \sigma(t)$ is a vector field along σ in the sense of the definition in Chapter 2.

Then, the directional derivative of Y in the direction X at p, is defined as,

$$D_X Y = \frac{d}{dt} Y \circ \sigma(t)|_{t=0} \,. \tag{5.1}$$

I.e., to compute $D_X Y$, restrict Y to σ to obtain a vector valued function of t, and differentiate with respect to t.



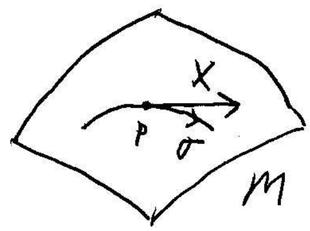
In terms of components, $Y = (Y^1, Y^2, Y^3)$,

$$\begin{aligned} D_X Y &= \frac{d}{dt} \left(Y^1 \circ \sigma(t), Y^2 \circ \sigma(t), Y^3 \circ \sigma(t)\right)|_{t=0} \\ &= \left(\frac{d}{dt} Y^1 \circ \sigma(t)|_{t=0}, \frac{d}{dt} Y^2 \circ \sigma(t)|_{t=0}, \frac{d}{dt} Y^3 \circ \sigma(t)|_{t=0}\right) \\ &= (D_X Y^1, D_X Y^2, D_X Y^3). \end{aligned}$$

Directional derivatives on surfaces.

Let M be a surface, and let $f : M \to I\!R$ be a smooth function on M. Recall, this means that $\hat{f} = f \circ \mathbf{x}$ is smooth for all proper patches $\mathbf{x} : U \to M$ in M.

Definition. For $p \in M$, $X \in T_pM$, the directional derivative of f at p in the direction X, denoted $\nabla_X f$, is defined as follows. Let $\sigma : (-\epsilon, \epsilon) \to M \subset I\!R^3$ be any smooth curve in M such that $\sigma(0) = p$ and $\sigma'(0) = X$:



Then,

$$\nabla_X f = \frac{d}{dt} f \circ \sigma(t)|_{t=0} \tag{5.2}$$

I.e., to compute $\nabla_X f$, restrict f to σ and differentiate with respect to parameter t.

Proposition 5.2. *The directional derivative is well-defined, i.e. independent of the particular choice of σ.*

Proof. Let $\mathbf{x} : U \to M$ be a proper patch containing p. Express σ in terms of coordinates in the usual manner,

$$\sigma(t) = \mathbf{x}(u^1(t), u^2(t)) .$$

By the chain rule,

$$\frac{d\sigma}{dt} = \sum \frac{du_i}{dt}\mathbf{x}_i \qquad \left(\mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u_i}\right) .$$

$X \in T_pM \Rightarrow X = \sum X^i \mathbf{x}_i$. The initial condition, $\dfrac{d\sigma}{dt}(0) = X$ then implies

$$\frac{du^i}{dt}(0) = X^i\,, \quad i = 1, 2.$$

Now,

$$\begin{aligned} f \circ \sigma(t) &= f(\sigma(t)) = f(\mathbf{x}(u^1(t), u^2(t)) \\ &= f \circ \mathbf{x}(u^1(t), u^2(t)) \\ &= \hat{f}(u^1(t), u^2(t)). \end{aligned}$$

Hence, by the chain rule,

$$\begin{aligned} \frac{d}{dt} f \circ \sigma(t) &= \frac{\partial \hat{f}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \hat{f}}{\partial u^2}\frac{du^2}{dt} \\ &= \sum_i \frac{\partial \hat{f}}{\partial u^i}\frac{du^i}{dt} = \sum_i \frac{du^i}{dt}\frac{\partial \hat{f}}{\partial u^i}. \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla_X f &= \frac{d}{dt} f \circ \sigma(t)|_{t=0} \\ &= \sum_i \frac{du^i}{dt}(0)\frac{\partial \hat{f}}{\partial u^i}(u^1, u^2)\,, \qquad (p = \mathbf{x}(u^1, u^2)) \\ \nabla_X f &= \sum_i X^i \frac{\partial \hat{f}}{\partial u^i}(u^1, u^2)\,, \end{aligned}$$

or simply,

$$\begin{aligned} \nabla_X f = \sum X^i \frac{\partial \hat{f}}{\partial u^i} \\ = X^1 \frac{\partial \hat{f}}{\partial u^1} + X^2 \frac{\partial \hat{f}}{\partial u^2}\,. \end{aligned} \tag{5.3}$$

$\square$

Example. Let $X = \mathbf{x}_1$. Since $\mathbf{x}_1 = 1 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2$, $\quad X^1 = 1$ and $X^2 = 0$. Hence the above equation implies, $\nabla_{\mathbf{x}_1} f = \dfrac{\partial \hat{f}}{\partial u^1}$. Similarly, $\nabla_{\mathbf{x}_2} f = \dfrac{\partial \hat{f}}{\partial u^2}$. I.e.,

$$\nabla_{\mathbf{x}_i} f = \frac{\partial \hat{f}}{\partial u^i}, \quad i = 1, 2. \tag{5.4}$$

The following proposition summarizes some basic properties of directional derivatives in surfaces.

Proposition 5.3.

(1) $\nabla_{(aX+bY)} f = a\nabla_X f + b\nabla_Y f$

(2) $\nabla_X (f + g) = \nabla_X f + \nabla_X g$

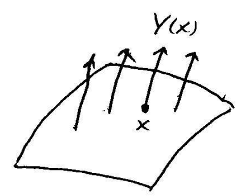
(3) $\nabla_X fg = (\nabla_X f)g + f(\nabla_X g)$

EXERCISE 5.1. Prove this proposition.

Vector fields along a surface.

A vector field along a surface M is a rule which assigns to each point of M a vector at that point,

$$x \in M \to Y(x) \in T_x\mathbb{R}^3$$



N.B. $\;Y(x)$ need not be tangent to M.

Analytically vector fields along a surface M are described by mappings.

$$Y : M \to \mathbb{R}^3$$
$$Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \in T_x\mathbb{R}^3$$

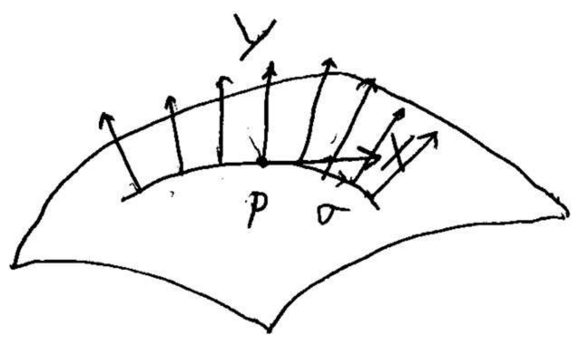
Components of Y: $\;Y^i : M \to \mathbb{R}$, $i = 1, 2, 3$. We say Y is *smooth* if its component functions are smooth.

The directional derivative of a vector field along M is defined in a manner similar to the directional derivative of a function defined on M.

Given a vector field along M, $Y : M \to I\!R^3$, for $p \in M, X \in T_pM$, the *directional derivative of* Y *in the direction* X, denoted $\nabla_X Y$, is defined as,

$$\nabla_X Y = \frac{d}{dt} Y \circ \sigma(t)|_{t=0}$$

where $\sigma : (-\epsilon, \epsilon) \to M$ is a smooth curve in M such that $\sigma(0) = p$ and $\dfrac{d\sigma}{dt}(0) = X$.



I.e. to compute $\nabla_X Y$, restrict Y to σ to obtain a vector valued function of t - then differentiate with respect to t.

Proposition 5.4. *If* $Y(x) = (Y^1(x), Y^2(x), Y^3(x))$ *then,*

$$\nabla_X Y = (\nabla_X Y^1, \nabla_X Y^2, \nabla_X Y^3)\,.$$

Proof. Exercise.

Surface Coordinate Expression. Let $\mathbf{x} : U \to M$ be a proper patch in M containing p. Let $X \in T_pM$, $X = \sum_i x^i \mathbf{x}_i$. An argument like that for functions on M shows,

$$\nabla_X Y = \sum_{i=1}^{2} X^i \frac{\partial \hat{Y}}{\partial u^i}(u^1, u^2)\,, \qquad (p = \mathbf{x}(u^1, u^2))$$

where $\hat{Y} = Y \circ \mathbf{x} : U \to I\!R^3$ is Y expressed in terms of coordinates.

EXERCISE 5.2. Derive the expression above for $\nabla_X Y$. In particular, show

$$\nabla_{\mathbf{x}_i} Y = \frac{\partial \hat{Y}}{\partial u^i}, \quad i = 1, 2. \tag{5.5}$$

Some basic properties are described in the following proposition.

Proposition 5.5.

(1) $\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$

(2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

(3) $\nabla_X (fY) = (\nabla_X f)Y + f\nabla_X Y$

(4) $\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

The Weingarten Map and the 2nd Fundamental Form.

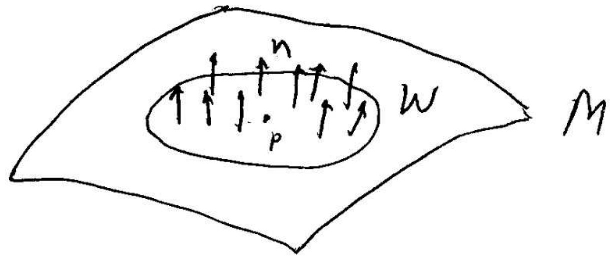
We are interested in studying the *shape* of surfaces in $I\!R^3$. Our approach (essentially due to Gauss) is to study how the unit normal to the surface "wiggles" along the surface.

The objects which describe the shape of M are:

1. The *Weingarten Map*, or *shape* operator. For each $p \in M$ this is a certain linear transformation $L : T_pM \to T_pM$.

2. The *second fundamental form*. This is a certain bilinear form $\mathcal{L} : T_pM \times T_pM \to I\!R$ associated in a natural way with the Weingarten map.

We now describe the Weingarten map. Fix $p \in M$. Let $n : W \to I\!R^3$, $p \in W \to n(p) \in T_pI\!R^3$, be a smooth *unit normal* vector field defined along a neighborhood W of p (see figure next page).



<u>Remarks</u>

1. n can always be constructed by introducing a proper patch $\mathbf{x} : U \to M$, $\mathbf{x} = \mathbf{x}(u^1, u^2)$ containing p:

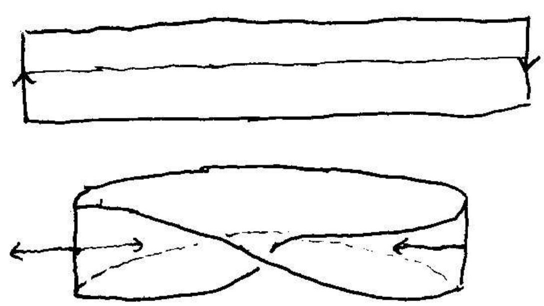
$$\hat{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|},$$

$\hat{n} : U \to I\!R^3$, $\hat{n} = \hat{n}(u^1, u^2)$. Then, $n = \hat{n} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to I\!R^3$ is a smooth unit normal v.f. along $\mathbf{x}(U)$.

2. The choice of n is not quite unique: $n \to -n$; choice of n is unique "up to sign"

3. A smooth unit normal field n always exists in a neighborhood of any given point p, but it may not be possible to extend n to all of M. This depends on whether or not M is an *orientable* surface.

Ex. Möbius band.



Lemma 5.6. *Let M be a surface, $p \in M$, and n be a smooth unit normal vector field defined along a neighborhood $W \subset M$ of p. Then for any $X \in T_pM$, $\nabla_X n \in T_pM$.*

Proof. It suffices to show that $\nabla_X n$ is perpendicular to n. $|n| = 1 \Rightarrow \langle n, n\rangle = 1 \Rightarrow$

$$\begin{aligned} \nabla_X \langle n, n\rangle &= \nabla_X 1 \\ \langle \nabla_X n, n\rangle + \langle n, \nabla_X n\rangle &= 0 \\ 2\langle \nabla_X n, n\rangle &= 0 \end{aligned}$$

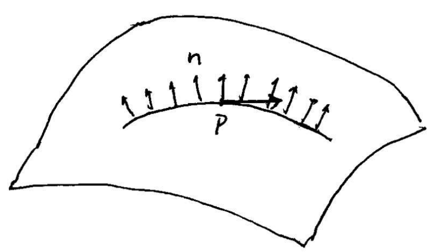
and hence $\nabla_X n \perp n$.

Definition. Let M be a surface, $p \in M$, and n be a smooth unit normal v.f. defined along a nbd $W \subset M$ of p. The *Weingarten Map* (or *shape operator*) is the map $L : T_pM \to T_pM$ defined by,

$$L(X) = -\nabla_X n\,.$$

<u>*Remarks*</u>

1. The minus sign is a convention – will explain later.
2. $L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0}$



Lemma 5.7. *$L : T_pM \to T_pM$ is a linear map, i.e.,*

$$L(aX + bY) = aL(X) + bL(Y)\,.$$

for all $X, Y, \in T_pM$, $a, b \in \mathbb{R}$.

Proof. This follows from properties of directional derivative,

$$\begin{aligned} L(aX + bY) &= -\nabla_{aX+bY} n \\ &= -[a\nabla_X n + b\nabla_Y n] \\ &= a(-\nabla_X n) + b(-\nabla_Y n) \\ &= aL(X) + bL(Y). \end{aligned}$$

Example. Let M be a *plane* in $I\!R^3$:

$$M : ax + by + cz = d\,.$$

Determine the Weingarten Map at each point of M. We note that unit normal is given by,

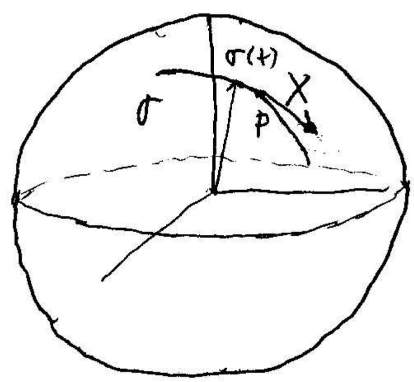
$$n = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} = \left(\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda}\right)\,,$$

where $\lambda = \sqrt{a^2+b^2+c^2}$. Hence,

$$\begin{aligned} L(X) &= -\nabla_X n = -\nabla_X \left(\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda}\right) \\ &= -\left(\nabla_X \frac{a}{\lambda}, \nabla_X \frac{b}{\lambda}, \nabla_X \frac{c}{\lambda}\right) = \mathbf{0}. \end{aligned}$$

Therefore $L(X) = 0 \ \forall \ \ X \in T_pM$, i.e. $\ L \equiv 0$.

Example. Let $M = S_r^2$ be the sphere of radius r, let n be the *outward* pointing unit normal. Determine the Weingarten map at each point of M.



Fix $p \in S_r^2$, and let $X \in T_pS_r^2$. Let $\sigma : (-\varepsilon, \varepsilon) \to S_r^2$ be a curve in S_r^2 such that $\sigma(0) = p$, $\dfrac{d\sigma}{dt}(0) = X$

Then,

$$L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0}\,.$$

But note, $n \circ \sigma(t) = n(\sigma(t)) = \dfrac{\sigma(t)}{|\sigma(t)|} = \dfrac{\sigma(t)}{r}$. Hence,

$$\begin{aligned} L(X) &= -\frac{d}{dt}\frac{\sigma(t)}{r}|_{t=0} = -\frac{1}{r}\frac{d\sigma}{dt}|_{t=0} \\ L(X) &= -\frac{1}{r}X\,. \end{aligned}$$

for all $X \in T_pM$. Hence,

$$L = -\frac{1}{r}id\,, \tag{5.6}$$

where $id : T_pM \to T_pM$ is the identity map, $id(X) = X$.

Remark: If we had taken the inward pointing normal then $L = \dfrac{1}{r}id$.

Definition. For each $p \in M$, the *second fundamental form* is the bilinear form
$\mathcal{L} : T_pM \times T_pM \to I\!R$ defined by,

$$\begin{aligned} \mathcal{L}(X,Y) &= \langle L(X), Y\rangle \\ &= -\langle \nabla_X n, Y\rangle\,. \end{aligned} \tag{5.7}$$

Note that $\mathcal{L}$ is indeed *bilinear*, e.g.,

$$\begin{aligned} \mathcal{L}(aX + bY, Z) &= \langle L(aX + bY), Z\rangle \\ &= \langle aL(X) + bL(Y), Z\rangle \\ &= a\langle L(X), Z\rangle + b\langle L(Y), Z\rangle \\ &= a\mathcal{L}(X,Z) + b\mathcal{L}(Y,Z). \end{aligned}$$

Example. If M is a plane, then $\mathcal{L} \equiv 0$:

$$\mathcal{L}(X,Y) = \langle L(X), Y\rangle = \langle 0, Y\rangle = 0\,.$$

Example. The sphere S_r^2 of radius r, $\mathcal{L} : T_pS_r^2 \times T_pS_r^2 \to I\!R$,

$$\begin{aligned} \mathcal{L}(X,Y) &= \langle L(X), Y\rangle \\ &= \langle -\frac{1}{r}X, Y\rangle \\ &= -\frac{1}{r}\langle X, Y\rangle \end{aligned}$$

Hence, $\mathcal{L} = -\frac{1}{r}\langle\ ,\ \rangle$. The second fundamental form is a multiple of the first fundamental form!

Coordinate expressions

Let $\mathbf{x} : U \to M$ be a patch containing $p \in M$. Then $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for T_pM. We express $L : T_pM \to T_pM$ and $\mathcal{L} : T_pM \times T_pM \to \mathbb{R}$ with respect to this basis. Since $L(\mathbf{x}_j) \in T_pM$, we have,

$$\begin{aligned} L(\mathbf{x}_j) &= L^1{}_j\mathbf{x}_1 + L^2{}_j\mathbf{x}_2\,, \qquad j = 1,2 \\ &= \sum_{i=1}^{2} L^i{}_j\mathbf{x}_i\,. \end{aligned} \tag{5.8}$$

The numbers $L^i{}_j$, $1 \le i, j \le 2$, are called the components of L with respect to the coordinate basis $\{\mathbf{x}_1, \mathbf{x}_2\}$. The 2×2 matrix $[L^i{}_j]$ is the matrix representing the linear map L with respect to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$.

EXERCISE 5.3. Let $X \in T_pM$ and let $Y = L(X)$. In terms of components, $X = \sum_j X^j\mathbf{x}_j$ and $Y = \sum_i Y^i\mathbf{x}_i$. Show that

$$Y^i = \sum_j L^i{}_jX^j\,, \qquad i = 1,2\,,$$

which in turn implies the matrix equation,

$$\begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} = \left[L^i{}_j\right] \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}.$$

This is the Weingarten map expressed as a matrix equation.

Introduce the unit normal field along $W = \mathbf{x}(U)$ with respect to the patch $\mathbf{x} : U \to M$,

$$\begin{aligned} \hat{n} &= \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}\,, \qquad \hat{n} = \hat{n}(u^1, u^2)\,, \\ n &= \hat{n} \circ \mathbf{x}^{-1} : W \to \mathbb{R}\,. \end{aligned}$$

Then by Exercise 5.2,

$$L(\mathbf{x}_j) = -\nabla_{\mathbf{x}_j} n = -\frac{\partial \hat{n}}{\partial u^j}\,.$$

Setting $n_j = \frac{\partial \hat{n}}{\partial u^j}$ and using (5.8), we obtain,

$$n_j = -L(\mathbf{x}_j)$$

$$n_j = -\sum_i L^i{}_j \mathbf{x}_i\,, \quad j = 1, 2\,. \tag{5.9}$$

These are known as the *Weingarten equations.* They can be used to compute the components of the Weingarten map. However, in practice it turns out to be more useful to have a formula for computing the components of the second fundamental form.

Components of $\mathcal{L}$*:* The components of $\mathcal{L}$ with respect to $\{\mathbf{x}_1, \mathbf{x}_2\}$ are defined as,

$$L_{ij} = \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j)\,, \qquad 1 \le i, j \le 2\,. \tag{5.10}$$

Using bilinearity, we sse that the components completely determine $\mathcal{L}$,

$$\begin{aligned} \mathcal{L}(X, Y) &= \mathcal{L}(\sum_i X^i \mathbf{x}_i, \sum_j Y^j \mathbf{x}_j) \\ &= \sum_{i,j} X^i Y^j \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) \\ &= \sum_{i,j} L_{ij} X^i Y^j\,. \end{aligned} \tag{5.11}$$

The following proposition provides a very useful formula for computing the L_{ij}'s.

Proposition 5.8. *The components* L_{ij} *of* $\mathcal{L}$ *are given by,*

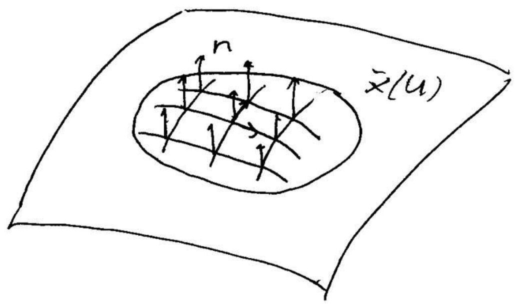
$$L_{ij} = \langle \hat{n}, \mathbf{x}_{ij} \rangle\,,$$

where $\mathbf{x}_{ij} = \dfrac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i}$.

Remark: Henceforth we no longer distinguish between n and $\hat{n}$, i.e., lets agree to drop the " $\hat{}$ ", then,

$$L_{ij} = \langle n, \mathbf{x}_{ij} \rangle\,, \tag{5.12}$$

Proof.



Along $\mathbf{x}(U)$ we have, $\langle n, \frac{\partial \mathbf{x}}{\partial u^j}\rangle = 0$, and hence,

$$\begin{aligned}
\frac{\partial}{\partial u^i}\langle n, \frac{\partial \mathbf{x}}{\partial u^j}\rangle &= 0 \\
\langle \frac{\partial n}{\partial u^i}, \frac{\partial \mathbf{x}}{\partial u^j}\rangle + \langle n, \frac{\partial^2 \mathbf{x}}{\partial u^i \partial u^j}\rangle &= 0 \\
\langle \frac{\partial n}{\partial u^i}, \frac{\partial \mathbf{x}}{\partial u^j}\rangle &= -\langle n, \frac{\partial^2 \mathbf{x}}{\partial u^i \partial u^j}\rangle = -\langle n, \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i}\rangle \,,
\end{aligned}$$

or, using shorthand notation,

$$\langle n_i, \mathbf{x}_j\rangle = -\langle n, \mathbf{x}_{ij}\rangle \,.$$

But,

$$\begin{aligned}
L_{ij} &= \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) = \langle L(\mathbf{x}_i), \mathbf{x}_j\rangle \\
&= -\langle n_i, \mathbf{x}_j\rangle \,,
\end{aligned}$$

and hence $L_{ij} = \langle n, \mathbf{x}_{ij}\rangle$. □

Observe,

$$\begin{aligned}
L_{ij} &= \langle n, \mathbf{x}_{ij}\rangle \\
&= \langle n, \mathbf{x}_{ji}\rangle \quad \text{(mixed partials equal!)} \\
L_{ij} &= L_{ji} \,, \qquad 1 \le i, j \le 2 \,. \qquad (5.13)
\end{aligned}$$

In other words, $\mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) = \mathcal{L}(\mathbf{x}_j, \mathbf{x}_i)$. In fact, using (5.11) and (5.13), one sees that this holds for all tangent vectors X, Y.

Proposition 5.9. *The second fundamental form $\mathcal{L} : T_pM \times T_pM \to I\!R$ is symmetric, i.e.*

$$\mathcal{L}(X, Y) = \mathcal{L}(Y, X) \;\; \forall \;\; X, Y \in T_pM \,.$$

Relationship between $L^i{}_j$ and L_{ij}

$$\begin{aligned} L_{ij} &= \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) = \mathcal{L}(\mathbf{x}_j, \mathbf{x}_i) \\ &= \langle L(\mathbf{x}_j), \mathbf{x}_i \rangle = \langle \sum_k L^k{}_j \mathbf{x}_k, \mathbf{x}_i, \rangle \\ &= \sum_k L^k{}_j \langle \mathbf{x}_k, \mathbf{x}_i \rangle \,, \end{aligned}$$

and hence,

$$L_{ij} = \sum_k g_{ik} L^k{}_j, \quad 1 \le i, j \le 2 \,. \tag{5.14}$$

Classical tensor jargon: L_{ij} obtained from $L^k{}_j$ by "lowering the index k with the metric". The equation above implies the matrix equation

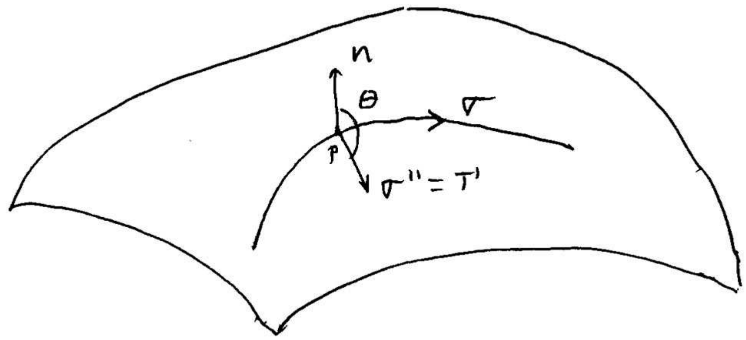
$$[L_{ij}] = [g_{ij}][L^i{}_j] \,. \tag{5.15}$$

Geometric Interpretation of the 2nd Fundamental Form

Normal Curvature. Let $s \to \sigma(s)$ be a unit speed curve lying in a surface M. Let p be a point on σ, and let n be a smooth unit normal v.f. defined in a nbd W of p. The *normal curvature* of σ at p, denoted κ_n, is defined to be the component of the curvature vector $\sigma'' = T'$ along n, i.e.,

$$\begin{aligned} \kappa_n &= \text{normal component of the curvature vector} \\ &= \langle \sigma'', n \rangle \\ &= \langle T', n \rangle \\ &= |T'|\,|n| \cos\theta \\ &= \kappa \cos\theta \,, \end{aligned}$$

where θ is the angle between the curvature vector T' and the surface normal n (see figure next page). If $\kappa \neq 0$ then, recall, we can introduce the principal normal N to σ, by the equation, $T' = \kappa N$; in this case θ is the angle beween N and n.



Remark: κ_n gives a measure of how much σ is bending in the direction perpendicular to the surface; it neglects the amount of bending tangent to the surface.

Proposition 5.10. *Let M be a surface, $p \in M$. Let $X \in T_pM$, $\quad |X| = 1$ (i.e. X is a* **unit** *tangent vector). Let $s \to \sigma(s)$ be any unit speed curve in M such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then*

$$\begin{aligned} \mathcal{L}(X,X) &= \text{normal curvature of } \sigma \text{ at } p \\ &= \kappa_n = \langle \sigma'', n \rangle. \end{aligned} \tag{5.16}$$

Proof. Along σ,

$$\begin{aligned} \langle \sigma'(s), n \circ \sigma(s) \rangle &= 0\,, \qquad \text{for all } s \\ \frac{d}{ds}\langle \sigma', n \circ \sigma \rangle &= 0 \\ \langle \sigma'', n \circ \sigma \rangle + \langle \sigma', \frac{d}{ds} n \circ \sigma \rangle &= 0\,. \end{aligned}$$

At $s = 0$,

$$\begin{aligned} \langle \sigma'', n \rangle + \langle X, \nabla_X n \rangle &= 0 \\ \langle \sigma'', n \rangle &= -\langle X, \nabla_X n \rangle \\ \kappa_n &= \langle X, L(X) \rangle \\ \kappa_n &= \langle L(X), X \rangle \\ &= \mathcal{L}(X,X). \end{aligned}$$

Remark: the sign convention used in the definition of the Weingarten map ensures that $\mathcal{L}(X,X) = +\kappa_n$ (rather than $-\kappa_n$).

Corollary 5.11. *All unit speed curves lying in a surface M which pass through $p \in M$ and have the same unit tangent vector X at p, have the same normal curvature at p. That is, the normal curvature depends only on the tangent direction X.*

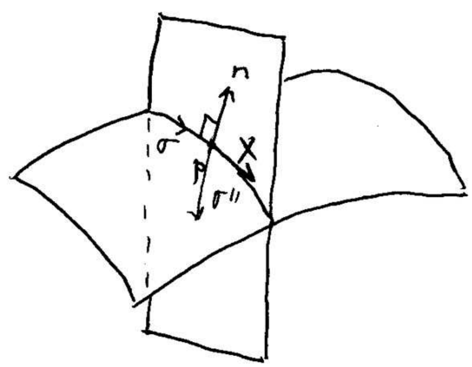
Thus it makes sense to say:

$$\mathcal{L}(X,X) \text{ is the normal curvature in the direction } X\,.$$

Given a unit tangent vector $X \in T_pM$, there is a distinguished curve in M, called the *normal section* at p in the direction X. Let,

$$\Pi = \text{plane through } p \text{ spanned by } n \text{ and } X.$$

Π cuts M in a curve σ. Parameterize σ wrt arc length, $s \to \sigma(s)$, such that $\sigma(0) = p$ and $\dfrac{d\sigma}{ds}(0) = X$:



By definition, σ is the *normal section* at p in the direction X. By the previous proposition,

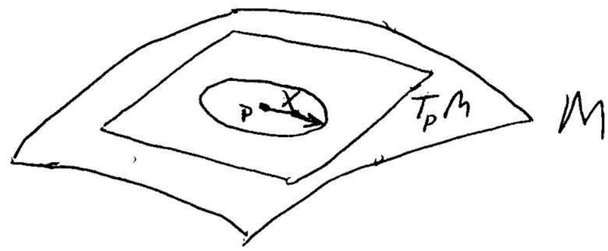
$$\begin{aligned}\mathcal{L}(X,X) &= \text{normal curvature of the normal section } \sigma\\ &= \langle \sigma'', n\rangle = \langle T', n\rangle\\ &= \kappa\cos\theta\,,\end{aligned}$$

where θ is the angle between n and T'. Since σ lies in Π, T' is tangent to Π, and since T' is also perpendicular to X, it follows that T' is a multiple of n. Hence, $\theta = 0$ or π, which implies that $\mathcal{L}(X,X) = \pm\kappa$.

Thus we conclude that,

$$\boxed{\mathcal{L}(X,X) = \textit{signed} \text{ curvature of the normal section at } p \text{ in the direction } X.}$$

Principal Curvatures.



The set of unit tangent vectors at p, $X \in T_pM$, $|X| = 1$, forms a circle in the tangent plane to M at p. Consider the function from this circle into the reals,

$$\begin{aligned} X &\to \text{normal curvature in direction } X \\ X &\to \mathcal{L}(X, X). \end{aligned}$$

The *principal curvatures* of M at p, $\kappa_1 = \kappa_1(p)$ and $\kappa_2 = \kappa_2(p)$, are defined as follows,

$$\begin{aligned} \kappa_1 &= \textit{the maximum normal curvature at } p \\ &= \max_{|X|=1} \mathcal{L}(X, X) \end{aligned}$$

$$\begin{aligned} \kappa_2 &= \textit{the minimum normal curvature at } p \\ &= \min_{|X|=1} \mathcal{L}(X, X) \end{aligned}$$

This is the *geometric* characterization of principal curvatures. There is also an important *algebraic* characterization.

Some Linear Algebra

Let V be a vector space over the reals, and let $\langle \ , \ \rangle : V \times V \to \mathbb{R}$ be an inner product on V; hence V is an inner product space. Let $L : V \to V$ be a linear transformation. Our main application will be to the case: $V = T_pM$, $\langle \ , \ \rangle$ = induced metric, and L = Weingarten map.

L is said to be *self adjoint* provided

$$\langle L(v), w \rangle = \langle v, L(w) \rangle \quad \forall \ v, w \in V \, .$$

Remark. Let $V = \mathbb{R}^n$, with the usual dot product, and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $[L^i{}_j]$ = matrix representing L with respect to the *standard* basis, $e_1 : (1, 0, ..., 0)$, etc. Then L is self-adjoint if and only if $[L^i{}_j]$ is *symmetric* $[L^i{}_j] = [L^j{}_i]$.

Proposition 5.12. *The Weingarten map $L : T_pM \to T_pM$ is self adjoint, i.e.*

$$\langle L(X), Y\rangle = \langle X, L(Y)\rangle \ \ \forall \ \ X, Y \in T_pM,$$

where $\langle\ ,\ \rangle$ = 1st fundamental form.

Proof. We have,

$$\begin{aligned}\langle L(X), Y\rangle &= \mathcal{L}(X, Y) = \mathcal{L}(Y, X) \\ &= \langle L(Y), X\rangle = \langle X, L(Y)\rangle .\end{aligned}$$

□

Self adjoint linear transformations have very nice properties, as we now discuss. For this discussion, we restrict attention to 2-dimensional vector spaces, $\dim V = 2$.

A vector $v \in V$, $v \neq 0$, is called an *eigenvector* of L if there is a real number λ such that,

$$L(v) = \lambda v\,.$$

λ is called an *eigenvalue* of L. The eigenvalues of L can be determined by solving

$$\det(A - \lambda I) = 0 \tag{5.17}$$

where A is a matrix representing L and I = identity matrix. The equation (5.17) is a quadratic equation in λ, and hence has at most 2 real roots; it may have no real roots.

Theorem 5.13 (Fundamental Theorem of Self Adjoint Operators)**.** *Let V be a 2-dimensional inner product space. Let $L : V \to V$ be a self-adjoint linear map. Then V admits an orthonormal basis consisting of eigenvectors of L. That is, there exists an orthonormal basis $\{e_1, e_2\}$ of V and real numbers λ_1, λ_2, $\lambda_1 \geq \lambda_2$ such that*

$$L(e_1) = \lambda_1 e_1, \quad L(e_2) = \lambda_2 e_2,$$

i.e., e_1 and e_2 are eigenvectors of L and λ_1, λ_2 are the corresponding eigenvalues. Moreover the eigenvalues are given by

$$\begin{aligned}\lambda_1 &= \max_{|v|=1}\langle L(v), v\rangle \\ \lambda_2 &= \min_{|v|=1}\langle L(v), v\rangle\,.\end{aligned}$$

Proof. A proof is given in Do Carmo [1], p. 214 f.

Remark on orthogonality of eigenvectors. Let e_1, e_2 be eigenvectors with eigenvalues λ_1, λ_2. If $\lambda_1 \neq \lambda_2$, then e_1 and e_2 are necessarily orthogonal, as seen by the following,

$$\lambda_1\langle e_1, e_2\rangle = \langle L(e_1), e_2\rangle = \langle e_1, L(e_2)\rangle = \lambda_2\langle e_1, e_2\rangle\,,$$

$\Rightarrow \quad (\lambda_1 - \lambda_2)\langle e_1, e_2\rangle = 0 \Rightarrow \langle e_1, e_2\rangle = 0$. On the other hand, if $\lambda_1 = \lambda_2 = \lambda$ then $L(v) = \lambda v$ for all v. Hence *any* o.n. basis is a basis of eigenvectors.

We now apply these facts to the Weingarten map,

$$\begin{aligned}
&L : T_pM \to T_pM\,,\\
&\mathcal{L} : T_pM \times T_pM \to I\!R, \quad \mathcal{L}(X,Y) = \langle L(X), Y\rangle\,.
\end{aligned}$$

Since L is self adjoint, and, by definition,

$$\begin{aligned}
\kappa_1 = \max_{|X|=1} \mathcal{L}(X,X) &= \max_{|X|=1}\langle L(X), X\rangle\\
\kappa_2 = \min_{|X|=1} \mathcal{L}(X,X) &= \min_{|X|=1}\langle L(X), X\rangle\,,
\end{aligned}$$

we obtain the following.

Theorem 5.14. *The principal curvatures κ_1, κ_2 of M at p are the eigenvalues of the Weingarten map $L : T_pM \to T_pM$. There exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that*

$$L(e_1) = \kappa_1 e_1, \quad L(e_2) = \kappa_2 e_2\,,$$

i.e., e_1, e_2 are eigenvectors of L associated with the eigenvalues κ_1, κ_2, respectively. The eigenvectors e_1 and e_2 are called **principal directions**.

Observe that,

$$\begin{aligned}
\kappa_1 &= \kappa_1\langle e_1, e_1\rangle = \langle L(e_1), e_1\rangle = \mathcal{L}(e_1, e_1)\\
\kappa_2 &= \kappa_2\langle e_2, e_2\rangle = \langle L(e_2), e_2\rangle = \mathcal{L}(e_2, e_2)\,,
\end{aligned}$$

i.e., the principal curvature κ_1 is the normal curvature in the principal direction e_1, and similarly for κ_2.

Now, let A be the matrix associated to the Weingarten map L with respect to the orthonormal basis $\{e_1, e_2\}$; thus,

$$\begin{aligned} L(e_1) &= \kappa_1 e_1 + 0 e_2 \\ L(e_2) &= 0 e_1 + \kappa_2 e_2 \end{aligned}$$

which implies,

$$A = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}.$$

Then,

$$\begin{aligned} \det L &= \det A = \kappa_1 \kappa_2 \\ \operatorname{tr} L &= \operatorname{tr} A = \kappa_1 + \kappa_2 \,. \end{aligned}$$

Definition. The *Gaussian curvature* of M at p, $K = K(p)$, and the *mean curvature* of M at p, $H = H(p)$ are defined as follows,

$$K = \det L = \kappa_1 \kappa_2 \tag{5.18}$$

$$H = \operatorname{tr} L = \kappa_1 + \kappa_2 \,. \tag{5.19}$$

Remarks. The Gaussian curvature is the more important of the two curvatures; it is what is meant by the *curvature* of a surface. A famous discovery by Gauss is that it is intrinsic – in fact can be computed in terms of the g_{ij}'s (This is not obvious!). The *mean curvature* (which has to do with minimal surface theory) is *not* intrinsic. This can be easily seen as follows. Changing the normal $n \to -n$ changes the sign of the Weingarten map,

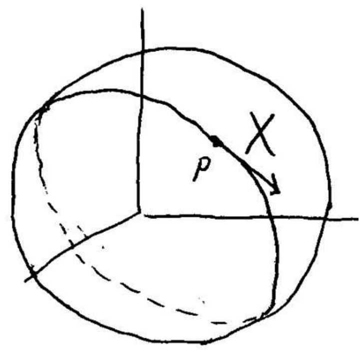
$$L_{-n} = -L_n \,.$$

This in turn changes the sign of the principal curvatures, hence $H = \kappa_1 + \kappa_2$ changes sign, but $K = \kappa_1 \kappa_2$ does not change sign.

Some Examples

Example. For S_r^2 = sphere of radius r, compute κ_1, κ_2, K, H (Use outward normal).

Geometrically: $p \in S_r^2$, $X \in T_pM$, $|X| = 1$,



$$\begin{aligned}\mathcal{L}(X,X) &= \pm \text{ curvature of normal section in direction } X\\ &= -\text{curvature of great circle}\\ &= -\frac{1}{r}.\end{aligned}$$

Therefore

$$\kappa_1 = \max_{|X|=1} \mathcal{L}(X,X) = -\frac{1}{r}\,,$$

$$\kappa_2 = \min_{|X|=1} \mathcal{L}(X,X) = -\frac{1}{r}\,,$$

$$K = \kappa_1\kappa_2 = \frac{1}{r^2} > 0, \quad H = \kappa_1 + \kappa_2 = -\frac{2}{r}\,.$$

Algebraically: We need to determine the eigenvalues of Weingarten map: $L : T_pM \to T_pM$. We showed previously,

$$\begin{aligned}L &= -\frac{1}{r} id, \quad \text{i.e.,}\\ L(X) &= -\frac{1}{r}X \quad \text{for all } X \in T_pM.\end{aligned}$$

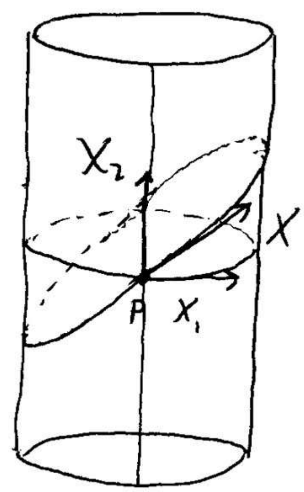
Thus, with respect to *any* orthonormal basis $\{e_1, e_2\}$ of T_pM,

$$L(e_i) = -\frac{1}{r}e_i \quad i = 1, 2.$$

Therefore, $\kappa_1 = \kappa_2 = -\frac{1}{r}, \;\; K = \frac{1}{r^2}, \;\; H = -\frac{2}{r}.$

Example. Let M be the cylinder of radius a: $x^2 + y^2 = a^2$. Compute κ_1, κ_2, K, H. (Use the inward pointing normal.)

<u>Geometrically:</u>



$$\begin{aligned}
\mathcal{L}(X_1, X_1) &= \pm \text{ curvature of normal section in direction } X_1 \\
&= + \text{ curvature of circle of radius } a \\
&= \frac{1}{a}\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(X_2, X_2) &= \pm \text{ curvature of normal section in direction } X_2 \\
&= \text{ curvature of line} \\
&= 0\,.
\end{aligned}$$

In general, for $X \neq X_1, X_2$,

$$\mathcal{L}(X,X) = \text{ curvature of ellipse through } p\,.$$

The curvature is between 0 and $\frac{1}{a}$, and thus,

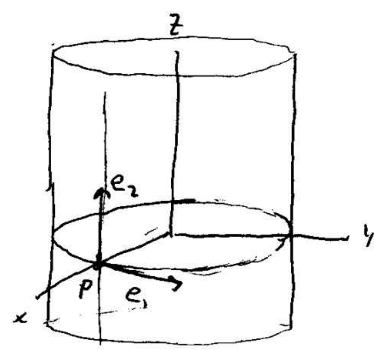
$$0 \leq \mathcal{L}(X,X) \leq \frac{1}{a}\,.$$

We conclude that,

$$\begin{aligned}
\kappa_1 &= \max_{|X|=1} \mathcal{L}(X,X) = \mathcal{L}(X_1,X_1) = \frac{1}{a}\,, \\
\kappa_2 &= \min_{|X|=1} \mathcal{L}(X,X) = \mathcal{L}(X_2,X_2) = 0\,.
\end{aligned}$$

Thus, $K = 0$ (cylinder is flat!) and $H = \dfrac{1}{a}$.

Algebraically: Determine the eigenvalues of the Weingarten map. By a rotation and translation we may take p to be the point $p = (a,0,0)$. Let $e_1, e_2 \in T_pM$ be the tangent vectors $e_1 = (0,1,0)$ and $e_2 = (0,0,1)$.



To compute $L(e_1)$, consider the circle,

$$\sigma(s) = (a\cos(\frac{s}{a}), a\sin(\frac{s}{a}), 0)$$

Note that $\sigma(0) = p$ and $\sigma'(0) = e_1$. Thus,

$$\begin{aligned}
L(e_1) &= -\nabla_{e_1} n \\
&= -\frac{d}{ds} n(\sigma(s))\Big|_{s=0}
\end{aligned}$$

But,

$$\begin{aligned} n(\sigma(s)) &= -\frac{\sigma(s)}{|\sigma(s)|} = -\frac{\sigma(s)}{a} \\ &= -\left(\cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right), 0\right) \end{aligned}$$

Therefore,

$$\begin{aligned} L(e_1) &= \frac{d}{ds}\left(\cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right), 0\right)|_{s=0} \\ &= \frac{1}{a}\left(-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right), 0\right)|_{s=0} \\ &= \frac{1}{a}(0,1,0) \\ L(e_1) &= \frac{1}{a}e_1 \end{aligned}$$

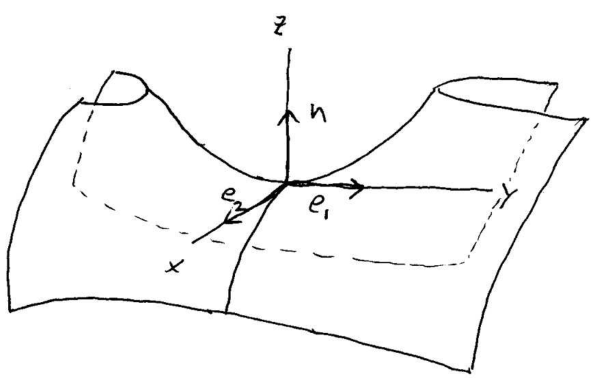
Thus, e_1 is an eigenvector with eigenvalue $\frac{1}{a}$. Similarly (exercise!),

$$L(e_2) = \mathbf{0} = 0 \cdot e_2$$

i.e., e_2 is an eigenvector with eigenvalue 0. (Note; e_2 is tangent to a vertical line in the surface, along which n is constant.)

We conclude that, $\kappa_1 = \dfrac{1}{a}$, $\kappa_2 = 0$, $K = 0$, $H = \dfrac{1}{a}$.

Example. Consider the saddle surface, M: $z = y^2 - x^2$, Compute κ_1, κ_2, K, H at $p = (0,0,0)$.



$$\begin{aligned}\mathcal{L}(e_1,e_1) &= \pm \text{ curvature of normal section in direction of } e_1 \\ &= + \text{ curvature of } z = y^2\end{aligned}$$

The curvature is given by,

$$\kappa = \frac{\left|\dfrac{d^2z}{dy^2}\right|}{\left[1+\left(\dfrac{dz}{dy}\right)^2\right]^{3/2}} = 2$$

and so, $\mathcal{L}(e_1,e_1) = 2$. Similarly, $\mathcal{L}(e_2,e_2) = -2$. Observe,

$$\mathcal{L}(e_2,e_2) \leq \mathcal{L}(X,X) \leq \mathcal{L}(e_1,e_1)$$

Therefore, $\kappa_1 = 2$, $\kappa_2 = -2$, $K = -4$, and $H = 0$ at $(0,0,0)$.

EXERCISE 5.4. For the saddle surface M above, consider the Weingarten map $L : T_pM \to T_pM$ at $p = (0,0,0)$. Compute $L(e_1)$ and $L(e_2)$ directly from the definition of the Weingarten map to show,

$$L(e_1) = 2e_1 \text{ and } L(e_2) = -2e_2\,.$$

Hence, -2 and 2 are the eigenvalues of L, which means $\kappa_1 = 2$ and $\kappa_2 = -2$.

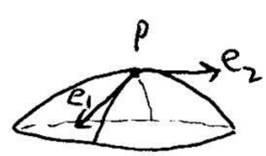
Remark. We have computed the quantities κ_1, κ_2, K, and H of the saddle surface only at a single point. To compute these quantities at all points, we will need to develop better computational tools.

Significance of the sign of Gaussian Curvature

We have,

$$K = \det L = \kappa_1\kappa_2.$$

1. $K > 0 \iff \kappa_1$ and κ_2 have the same sign $\iff$ the normal sections in the principal directions e_1, e_2 both bend in the same direction,



$K > 0$ at p.

Ex. $z = ax^2 + by^2$, a, b have the same sign (elliptic paraboloid). At $p = (0,0,0)$, $K = 4ab > 0$.

2. $K < 0 \iff \kappa_1$ and κ_2 have opposite signs $\iff$ normal sections in principle directions e_1 and e_2 bend in opposite directions,

$K < 0$ at p.

Ex. $z = ax^2 + by^2$, a, b have opposite sign (hyperbolic paraboloid). At $p = (0,0,0)$, $K = 4ab < 0$.

Thus, roughly speaking,

$$K > 0 \text{ at } p \Rightarrow \text{surface is "bowl-shaped" near } p$$
$$K < 0 \text{ at } p \Rightarrow \text{surface is "saddle-shaped" near } p$$

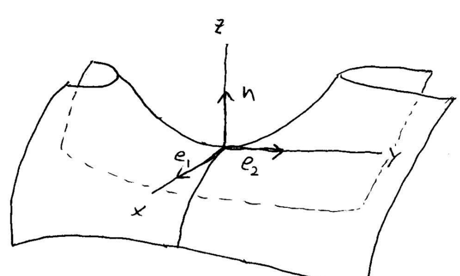
This rough observation can be made more precise, as we now show. Let M be a surface, $p \in M$. Let e_1, e_2 be principal directions at p. Choose e_1, e_2 so that $\{e_1, e_2, n\}$ is a positively oriented orthonornal basis.

By a translation and rotation of the surface, we can assume, (see the figure),

(1) $p = (0,0,0)$

(2) $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $n = (0,0,1)$ at p

(3) Near $p = (0,0,0)$, the surface can be described by an equation of form, $z = f(x,y)$, where $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth and $f(0,0) = 0$.



Proposition 5.15.

$$z = \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + \text{ \textit{higher order terms}}$$

Proof. Consider the Taylor series about $(0,0)$ for functions of two variables,

$$\begin{aligned} z = \;& f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2 \\ & + \text{ higher order terms}\,. \end{aligned}$$

We must compute 1st and 2nd order partial derivatives of f at $(0,0)$. Introduce the Monge patch, $\mathbf{x}(u,v) = (u, v, f(u,v))$, i.e,

$$\mathbf{x}: \begin{array}{l} x = u \\ y = v \\ z = f(u,v) \end{array}$$

We have,

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_u = (1, 0, f_u)\,, \\ \mathbf{x}_2 &= \mathbf{x}_v = (0, 1, f_v)\,, \\ n &= \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} \\ &= \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}} \end{aligned}$$

At $(u,v) = (0,0)$: $n = (0,0,1) \Rightarrow f_u = f_v = 0, \quad \Rightarrow \quad \mathbf{x}_1 = (1,0,0) = e_1$ and $\mathbf{x}_2 = (0,1,0) = e_2$.

Recall, the components of the 2nd fundamental form $L_{ij} = \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j)$ may be computed from the formula,

$$L_{ij} = \langle n, \mathbf{x}_{ij} \rangle, \quad \mathbf{x}_{ij} = \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i}\,.$$

In particular, $L_{11} = \langle n, \mathbf{x}_{11} \rangle$, where $\mathbf{x}_{11} = \mathbf{x}_{uu} = (0, 0, f_{uu})$.

At $(u,v) = (0,0)$: $\quad L_{11} = \langle n, \mathbf{x}_{11} \rangle = (0,0,1) \cdot (0, 0, f_{uu}(0,0)) = f_{uu}(0,0)\,.$

Therefore,

$$f_{uu}(0,0) = L_{11} = \mathcal{L}(\mathbf{x}_1, \mathbf{x}_1) = \mathcal{L}(e_1, e_1) = \kappa_1\,.$$

Similarly,

$$
\begin{aligned}
f_{vv}(0,0) &= \mathcal{L}(e_2,e_2) = \kappa_2 \\
f_{uv}(0,0) &= \mathcal{L}(e_1,e_2) = \langle L(e_1), e_2\rangle = \lambda_1(e_1,e_2) = 0.
\end{aligned}
$$

Thus, setting $x = u$, $y = v$, we have shown,

$$
\begin{aligned}
f_x(0,0) &= f_y(0,0) = 0 \\
f_{xx}(0,0) &= \kappa_1, \quad f_{yy}(0,0) = \kappa_2, \quad f_{xy}(0,0) = 0,
\end{aligned}
$$

which, substituting into the Taylor expansion, implies,

$$
z = \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + \text{ higher order terms.}
$$

$\square$

Computational Formula for Gaussian Curvature.

We now derive a useful expression for computing the Guassian curvature of more general surfaces.

We have,

$$
\begin{aligned}
K &= \text{Gaussian curvature} \\
&= \det L = \det[L^i{}_j].
\end{aligned}
$$

From equation (5.15) on p. 101,

$$
\begin{aligned}
[L_{ij}] &= [g_{ij}][L^i{}_j], \\
\det[L_{ij}] &= \det[g_{ij}]\det[L^i{}_j] \\
&= \det[g_{ij}] \cdot K
\end{aligned}
$$

Hence,

$$
K = \frac{\det[L_{ij}]}{\det[g_{ij}]}, \qquad g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j\rangle, \quad L_{ij} = \langle n, \mathbf{x}_{ij}\rangle .
$$

Further,

$$
\begin{aligned}
\det[L_{ij}] &= \det\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22}\end{bmatrix} \\
&= L_{11}L_{22} - L_{12}^2 ,
\end{aligned}
$$

since $L_{12} = L_{21}$, and similarly,

$$\det[g_{ij}] = g_{11}, g_{22} - g_{12}^2 .$$

Thus,

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} . \tag{5.20}$$

Example. Compute the Gaussian curvature of the saddle surface $z = y^2 - x^2$.

Introduce the Monge patch, $\mathbf{x}(u, v) = (u, v, v^2 - u^2)$.

<u>Compute the metric components g_{ij}:</u>

$$\begin{aligned} \mathbf{x}_u &= (1, 0, -2u), \quad \mathbf{x}_v = (0, 1, 2v) , \\ g_{uu} &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = (1, 0, -2u) \cdot (1, 0, -2u) \\ &= 1 + 4u^2. \end{aligned}$$

Similarly,

$$\begin{aligned} g_{vv} &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + 4v^2 , \\ g_{uv} &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = -4uv . \end{aligned}$$

Thus,

$$\begin{aligned} \det[g_{ij}] &= g_{uu}g_{vv} - g_{uv}^2 \\ &= (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 \\ &= 1 + 4u^2 + 4v^2. \end{aligned}$$

<u>Compute the second fundamental form components L_{ij}:</u>

We use, $L_{ij} = \langle n, \mathbf{x}_{ij} \rangle$. We have,

$$n = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}} ,$$

and,

$$\mathbf{x}_{uu} = (0, 0, -2), \quad \mathbf{x}_{vv} = (0, 0, 2), \quad \mathbf{x}_{uv} = (0, 0, 0) .$$

Then,

$$\begin{aligned} L_{uu} &= \langle n, \mathbf{x}_{uu} \rangle = \frac{-2}{\sqrt{1+4u^2+4v^2}} \\ L_{vv} &= \langle n, \mathbf{x}_{vv} \rangle = \frac{2}{\sqrt{1+4u^2+4v^2}} \\ L_{uv} &= \langle n, \mathbf{x}_{uv} \rangle = 0. \end{aligned}$$

Thus,

$$\det[L_{ij}] = L_{uu}L_{vv} - L_{uv}^2 = \frac{-4}{1+4u^2+4v^2} \,,$$

and therefore,

$$K(u,v) = \frac{\det[L_{ij}]}{\det[g_{ij}]} = \frac{-4}{1+4u^2+4v^2} \cdot \frac{1}{1+4u^2+4v^2}$$

$$K(u,v) = \frac{-4}{(1+4u^2+4v^2)^2}$$

Hence the saddle surface $z = y^2 - x^2$ has Gaussian curvature function,

$$K(x,y) = \frac{-4}{(1+4x^2+4y^2)^2} \,.$$

Observe that $K < 0$ everywhere, and $K = \dfrac{-4}{(1+4r^2)^2} \sim \dfrac{1}{r^4}$, where $r = \sqrt{x^2+y^2}$ is the distance from the z-axis. As $r \to \infty$, $K \to 0$ rapidly.

EXERCISE 5.5. Consider the surface M which is the graph of $z = f(x,y)$. Show that the Gaussian curvature $K = K(x,y)$ is given by,

$$K(x,y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}$$

where $f_x = \dfrac{\partial f}{\partial x}$, $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, etc.

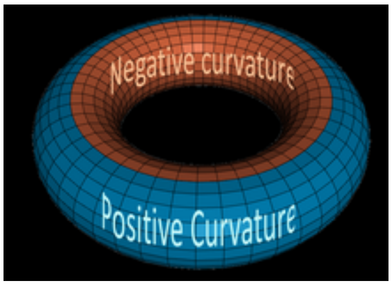
EXERCISE 5.6. Let M be the torus of large radius R and small radius r described in Exercise 3.3. Using the parameterization,

$$\mathbf{x}(t,\theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t)$$

show that the Gaussian curvature $K = K(t,\theta)$ is given by,

$$K = \frac{\cos t}{r(R + r\cos t)}.$$

Where on the torus is the Gaussian curvature negative? Where is it positive? (See figure.)



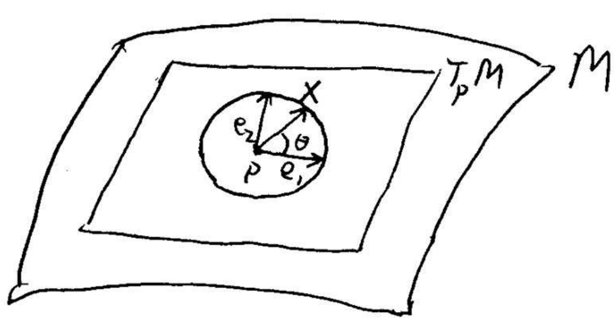
EXERCISE 5.7. Derive the following expression for the mean curvature H,

$$H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11}}{g_{11}g_{22} - g_{12}^2}.$$

The principal curvatures κ_1 and κ_2 at a point $p \in M$ are the normal curvatures in the principal directions e_1 and e_2. The normal curvature in any direction X is determined by κ_1 and κ_2 as follows.

If $X \in T_pM$, $|X| = 1$ then X can be expressed as (see the figure),

$$X = \cos\theta e_1 + \sin\theta e_2\,.$$



Proposition 5.16 (Euler's formula)**.** *The normal curvature in the direction X is given by,*

$$\mathcal{L}(X,X) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\,,$$

where κ_1, κ_2 are the principal curvatures, and θ is the angle between X and the principal direction e_1.

Proof. Use the shorthand, $c = \cos\theta,\ s = \sin\theta$. Then $X = ce_1 + se_2$, and

$$\begin{aligned} L(X) &= L(ce_1 + se_2) \\ &= cL(e_1) + sL(e_2) \\ &= c\kappa_1 e_1 + s\kappa_2 e_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}(X,X) &= \langle L(X), X\rangle \\ &= \langle c\kappa_1 e_1 + s\kappa_2 e_2, ce_1 + se_2\rangle \\ &= c^2\kappa_1 + s^2\kappa_2. \end{aligned}$$

EXERCISE 5.8. Assuming $\kappa_1 > \kappa_2$, determine where (i.e., for which values of θ) the function,

$$\kappa(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta, \quad 0 \le \theta \le 2\pi$$

achieves its maximum and minimum. The answer shows that the principal directions e_1, e_2 are unique, up to sign, in this case.

Gauss Theorema Egregium

The Weingarten map,

$$L(X) = -\nabla_X n$$

is an *extrinsically* defined object - it involves the normal to the surface. There is no reason to suspect that the determinant of L, the Gaussian curvature, is *intrinsic*, i.e. can be computed from measurements taken in the surface. But Gauss carried out some courageous computations and made the extraordinary discovery that, in fact, the Gaussian curvature K *is* intrinsic - i.e., can be computed from the g_{ij}'s. This is the most important result in the subject - albeit not the prettiest! If this result were not true then the subject of differential geometry, as we know it, would not exist.

We now embark on the same path - courageously carrying out the same computation.

Some notation. Introduce the "inverse" metric components, g^{ij}, $1 \le i, j \le 2$, by

$$[g^{ij}] = [g_{ij}]^{-1}\,,$$

i.e. g^{ij} is the i-jth entry of the inverse of the matrix $[g_{ij}]$. Using the formula,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\,,$$

we can express g^{ij} explicitly in terms of the g_{ij}, e.g.,

$$g^{11} = \frac{g_{22}}{g_{11}g_{22} - g_{12}^2}\,, \quad \text{etc.}$$

Note, in an orthogonal coordinate system, i.e., a proper patch in which $g_{12} = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0$, we have simply,

$$g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{12} = g^{21} = 0\,.$$

By the definition of an iinverse matrix, we have

$$[g_{ij}][g^{ij}] = I$$

where I = identity matrix = $[\delta_i{}^j]$. Here $\delta_i{}^j$ is the Kronecker delta (cf., equation (2.11) in Chapter 2),

$$\delta_i{}^j = \begin{cases} 0 & , i \ne j \\ 1, & i = j\,, \end{cases}$$

and so,

$$[g_{ij}][g^{ij}] = [\delta_i{}^j]\,.$$

The product formula for matrices then implies,

$$\sum_k g_{ik} g^{kj} = \delta_i{}^j$$

or, by the Einstein summation convention,

$$g_{ik}g^{kj} = \delta_i^j \,.$$

Now, let M be a surface and $\mathbf{x} : U \subset I\!R^2 \to M \subset I\!R^3$ be any proper patch in M. Then,

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}(u^1, u^2) = (x(u^1,u^2), y(u^1,u^2), z(u^1,u^2))\,, \\
\mathbf{x}_i &= \frac{\partial \mathbf{x}}{\partial u^i} = \left(\frac{\partial x}{\partial u^i}, \frac{\partial y}{\partial u^i}, \frac{\partial z}{\partial u^i}\right)\,, \\
\mathbf{x}_{ij} &= \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i} = \left(\frac{\partial^2 x}{\partial u^j \partial u^i}, \frac{\partial^2 y}{\partial u^j \partial u^i}, \frac{\partial^2 z}{\partial u^j \partial u^i}\right)\,.
\end{aligned}$$

We seek useful expressions for these second derivatives. At any point $p \in \mathbf{x}(U)$, $\{\mathbf{x}_1, \mathbf{x}_2, n\}$ form a basis for $T_p I\!R^3$. Since at p, $\mathbf{x}_{ij} \in T_p I\!R^3$, we can write,

$$\begin{aligned}
\mathbf{x}_{ij} &= \Gamma_{ij}^1 \mathbf{x}_1 + \Gamma_{ij}^2 \mathbf{x}_2 + \lambda_{ij} n\,, \\
\mathbf{x}_{ij} &= \sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \mathbf{x}_{\ell} + \lambda_{ij} n\,.
\end{aligned}$$

or, making use of the Einstein summation convention,

$$\mathbf{x}_{ij} = \Gamma_{ij}^{\ell} \mathbf{x}_{\ell} + \lambda_{ij} n\,. \tag{5.21}$$

We obtain expressions for $\lambda_{ij}, \Gamma_{ij}^{\ell}$. Dotting (5.21) with n gives,

$$\begin{aligned}
\langle \mathbf{x}_{ij}, n\rangle &= \Gamma_{ij}^{\ell}\langle \mathbf{x}_{\ell}, n\rangle + \lambda_{ij}\langle n, n\rangle \\
\Rightarrow \quad \lambda_{ij} &= \langle \mathbf{x}_{ij}, n\rangle = \langle n, \mathbf{x}_{ij}\rangle \\
\lambda_{ij} &= L_{ij}\,.
\end{aligned}$$

Dotting (5.21) with $\mathbf{x}_k$ gives,

$$\begin{aligned}
\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle &= \Gamma_{ij}^{\ell}\langle \mathbf{x}_{\ell}, \mathbf{x}_k\rangle + \lambda_{ij}\langle n, \mathbf{x}_k\rangle \\
\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle &= \Gamma_{ij}^{\ell} g_{\ell k}\,.
\end{aligned}$$

Solving for Γ^ℓ_{ij},

$$\begin{aligned}
\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle g^{km} &= \Gamma^\ell_{ij} g_{\ell k} g^{km} \\
&= \Gamma^\ell_{ij}\delta^m_\ell \\
\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle g^{km} &= \Gamma^m_{ij}
\end{aligned}$$

Thus,

$$\Gamma^\ell_{ij} = g^{k\ell}\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle\,. \tag{5.22}$$

Claim. The quantity $\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle$ is given by,

$$\begin{aligned}
\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle &= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \\
&= \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})
\end{aligned}.$$

Proof of the claim. We use Gauss' trick of permuting indices.

$$\begin{aligned}
g_{ij,k} &= \frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k}\langle \mathbf{x}_i, \mathbf{x}_j\rangle \\
&= \langle \frac{\partial \mathbf{x}_i}{\partial u^k}, \mathbf{x}_j\rangle + \langle \mathbf{x}_i, \frac{\partial \mathbf{x}_j}{\partial u^k}\rangle \\
(1) \quad & g_{ij,k} = \langle \mathbf{x}_{ik}, \mathbf{x}_j\rangle + \langle \mathbf{x}_i, \mathbf{x}_{jk}\rangle \\
(j \leftrightarrow k) \quad (2) \quad & g_{ik,j} = \langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle + \langle \mathbf{x}_i, \mathbf{x}_{kj}\rangle \\
(i \leftrightarrow j) \quad (3) \quad & g_{jk,i} = \langle \mathbf{x}_{ji}, \mathbf{x}_k\rangle + \langle \mathbf{x}_j, \mathbf{x}_{ki}\rangle
\end{aligned}$$

Then $(2) + (3) - (1)$ gives:

$$g_{ik,j} + g_{jk,i} - g_{ij,k} = 2\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle\,.$$

Thus we arrive at,

$$\Gamma^\ell_{ij} = \frac{1}{2} g^{k\ell}(g_{ik,j} + g_{jk,i} - g_{ij,k})\,. \tag{5.23}$$

Remark. These are known as the *Christoffel* symbols.

Summarizing, we have,

$$\mathbf{x}_{ij} = \Gamma^{\ell}_{ij}\mathbf{x}_{\ell} + L_{ij}n, \qquad \textit{(Gauss Formulas)} \tag{5.24}$$

where L_{ij} are the components of the 2nd fundamental form and Γ^{ℓ}_{ij} are the Christoffel symbols as given above.

Consider now the Gauss formulas together with the *Weingarten equations* (5.9),

$$n_j = -L^i{}_j\mathbf{x}_i\,.$$

We remark that the vector fields $\mathbf{x}_1, \mathbf{x}_2, n$, play a role in surface theory roughly analogous to the Frenet frame for curves. The Gauss formulas and Weingarten equations for the partial derivatives of $\mathbf{x}_1, \mathbf{x}_2, n$ then play a role roughly analogous to the Frenet formulas.

Now, Gauss takes things one step further and computes the 3rd derivatives, $\mathbf{x}_{ijk} = \dfrac{\partial}{\partial u^k}\mathbf{x}_{ij}$:

$$\begin{aligned}
\mathbf{x}_{ijk} &= \frac{\partial}{\partial u^k}(\Gamma^{\ell}_{ij}\mathbf{x}_{\ell} + L_{ij}n) = \frac{\partial}{\partial u^k}\Gamma^{\ell}_{ij}\mathbf{x}_{\ell} + \frac{\partial}{\partial u^k}L_{ij}n \\
&= \Gamma^{\ell}_{ij,k}\mathbf{x}_{\ell} + \Gamma^{\ell}_{ij}\mathbf{x}_{\ell k} + L_{ij,k}n + L_{ij}n_k \\
&= \Gamma^{\ell}_{ij,k}\mathbf{x}_{\ell} + \Gamma^{\ell}_{ij}(\Gamma^{m}_{\ell k}\mathbf{x}_m + L_{\ell k}n) + L_{ij,k}n + L_{ij}(-L^{\ell}{}_k\mathbf{x}_{\ell}) \\
&= \Gamma^{\ell}_{ij,k}\mathbf{x}_{\ell} + \underbrace{\Gamma^{\ell}_{ij}\Gamma^{m}_{\ell k}\mathbf{x}_m}_{\Gamma^{m}_{ij}\Gamma^{\ell}_{mk}\mathbf{x}_{\ell}} + \Gamma^{\ell}_{ij}L_{\ell k}n + L_{ij,k}n - L_{ij}L^{\ell}{}_k\mathbf{x}_{\ell}\,.
\end{aligned}$$

Thus,

$$\mathbf{x}_{ijk} = (\Gamma^{\ell}_{ij,k} + \Gamma^{m}_{ij}\Gamma^{\ell}_{mk} - L_{ij}L^{\ell}{}_k)\mathbf{x}_{\ell} + (L_{ij,k} + \Gamma^{\ell}_{ij}L_{\ell k})n\,,$$

and interchanging indices ($j \leftrightarrow k$),

$$\mathbf{x}_{ikj} = (\Gamma^{\ell}_{ik,j} + \Gamma^{m}_{ik}\Gamma^{\ell}_{mj} - L_{ik}L^{\ell}{}_j)\mathbf{x}_{\ell} + (L_{ik,j} + \Gamma^{\ell}_{ik}L_{\ell j})n\,.$$

Now, $\mathbf{x}_{ikj} = \mathbf{x}_{ijk}$ implies

$$\Gamma^{\ell}_{ik,j} + \Gamma^{m}_{ik}\Gamma^{\ell}_{mj} - L_{ik}L^{\ell}{}_j = \Gamma^{\ell}_{ij,k} + \Gamma^{m}_{ij}\Gamma^{\ell}_{mk} - L_{ij}L^{\ell}{}_k =$$

or,

$$\underbrace{\Gamma^{\ell}_{ik,j} - \Gamma^{\ell}_{ij,k} + \Gamma^{m}_{ik}\Gamma^{\ell}_{mj} - \Gamma^{m}_{ij}\Gamma^{\ell}_{mk}}_{R^{\ell}_{ijk}} = L_{ik}L^{\ell}{}_j - L_{ij}L^{\ell}{}_k\,.$$

These are the components of the famous *Riemann curvature tensor*. Note that thes componets are *intrinsic*, i.e. they can be computed from the g_{ij}'s (they involve 1st and 2nd derivatives of the g_{ij}'s).

We arrive at,

$$R^{\ell}_{ijk} = L_{ik}L^{\ell}{}_{j} - L_{ij}L^{\ell}{}_{k} \qquad \textit{The Gauss Equations}. \tag{5.25}$$

Theorem 5.17 (Gauss' Theorem Egregium)**.** *The Gaussian curvature of a surface is intrinsic, i.e. can be computed in terms of the g_{ij}'s.*

Proof. This follows from the Gauss equations. Multiply both sides by $g_{m\ell}$,

$$g_{m\ell}R^{\ell}_{ijk} = L_{ik}g_{m\ell}L^{\ell}{}_{j} - L_{ij}g_{m\ell}L^{\ell}{}_{k}.$$

But recall (see p. 13),

$$L_{mj} = g_{m\ell}L^{\ell}{}_{j}\,.$$

Hence,

$$g_{m\ell}R^{\ell}_{ijk} = L_{ik}L_{mj} - L_{ij}L_{mk}\,.$$

Setting $i = k = 1$, $m = j = 2$ we obtain,

$$\begin{aligned} g_{2\ell}R^{\ell}_{121} &= L_{11}L_{22} - L_{12}L_{21} \\ &= \det[L_{ij}]\,. \end{aligned}$$

Thus,

$$\begin{aligned} K &= \frac{\det[L_{ij}]}{\det[g_{ij}]} \\ K &= \frac{g_{2\ell}R^{\ell}_{121}}{g}, \qquad g = \det[g_{ij}] \end{aligned}$$

□

Comment. Gauss' Theorema Egregium can be interpreted in a slightly different way in terms of *isometries*. We discuss this point here very *briefly* and very *informally*.

Let M and N be two surfaces. A one-to-one, onto map $f : M \to N$ that preserves lengths of curves is called an isometry. (This may be understood at the level of tangent vectors: f takes curves to curves, and hence velocity vectors to velocity vectors. f is an isometry iff it preserves the length of velocity vectors iff it preserves the induced metrics g_M and g_N.) For example, the process of wrapping a piece of paper into a cylinder is an isometry.

Theorem 5.18. *Gaussian curvature is a bending invariant, i.e. is invariant under isometries, by which we mean: if $f : M \to N$ is an isometry then*

$$K_N(f(p)) = K_M(p)\,,$$

i.e., the Gaussian curvature is the same at corresponding points.

Proof. f preserves lengths and angles. Hence, in appropriate coordinate systems, the metric components for M and N are the same. By the formula for K above, the Gaussian curvature will be the same at corresponding points.

Application 1. The cylinder has Gaussian curvature $K = 0$ (because a plane has zero Gaussian curvature).

Application 2. No piece of a sphere can be fllattened into a piece of a plane without distorting distances (because $K_{\text{plane}} = 0$, $K_{\text{sphere}} = \dfrac{1}{r^2}$, r =radius).

Theorem (Riemann). Let M be a surface with vanishing Gaussian curvature, $K = 0$. Then each $p \in M$ has a neighborhood which is isometric to an open set in the Euclidean plane.

EXERCISE 5.9. Although The Gaussian curvature K is a "bending invariant", show that the principal curvatures κ_1, κ_2 are not. I.e., show that the principle curvatures are not in general invariant under an isometry. (Hint: Consider the bending of a rectangle into a cylinder).

Remark. It can be shown that in an orthogonal coordinate system,

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0\,,$$

the formula for the Gaussian curvature simplifies to the following

$$K = -\frac{1}{2\sqrt{g}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial u^2}\right)\right] \tag{5.26}$$

Additional Chapter 5 Exercises

1. Compute the Gaussian curvature K of the helicoid,

$$\mathbf{x}(u,v) = (u\cos v, u\sin v, bv)\,,$$

(where b is a positive constant). Show that,

$$K = -\frac{b^2}{(b^2+u^2)^2}$$

2. Let $u \to \sigma(u)$, $a < u < b$, be a regular curve in space, without self-intersections, and let p be a fixed point in space not on σ. The parameterized surface (called a *generalized cone*),

$$\mathbf{x}(u,v) = p + v\sigma(u), \qquad a < u < b,\ 0 < v,$$

describes a family of rays emanating from the point p.

   (a) Show that $\mathbf{x}$ is regular if and only if $\sigma(u) \times \sigma'(u) \neq 0$, for all $a < u < b$.

   (b) Assuming regularity, show by a computation that the surface is *flat*, i.e., has vanishing Gaussian curvature.

3. Let $\mathbf{x} = \mathbf{x}(u^1,u^2)$ be a proper patch in a surface M, and let $n = n(u^1,u^2)$, $n = \mathbf{x}_1 \times \mathbf{x}_2/|\mathbf{x}_1 \times \mathbf{x}_2|$, be the associated unit normal to M. Show that $n_1 \times n_2 = K\sqrt{g}\,n$, where $n_j = \frac{\partial n}{\partial u^j}$ and K is Gaussian curvature.

4. Suppose that in a proper patch $\mathbf{x} : U \to M$, $\mathbf{x} = \mathbf{x}(u,v)$, the Weingarten map satisfies $L = fI$, where $f = f(u,v)$ and $I =$ the identity map. Prove that f is constant. (Hint: Show that $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0$. To show this, consider L applied to the coordinate vectors.)

Chapter 6

Geodesics and the Gauss-Bonnet Theorem

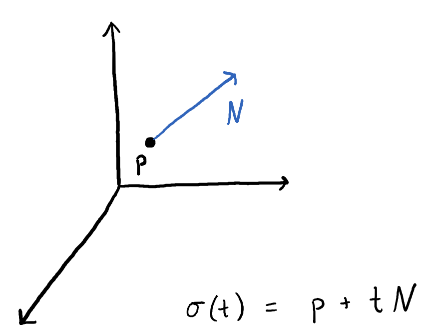
<u>**Geodesics in Surfaces**</u>.

We want to generalize the idea of a straight line in Euclidean space to surfaces. These generalized lines will be called *geodesics*. Before defining geodesics, let us first understand some properties of straight lines in $\mathbb{R}^3$.

(1) **Straight lines are curves of zero acceleration.** Consider a point $p \in \mathbb{R}^3$ and a vector N at p. We can generate the straight line σ at p in the direction N by defining $\sigma(t) = p + tN$. See the figure below. Then we have

$$\frac{d\sigma}{dt} = N \quad \text{and} \quad \frac{d^2\sigma}{dt^2} = 0.$$

Therefore straight lines are curves with zero acceleration.

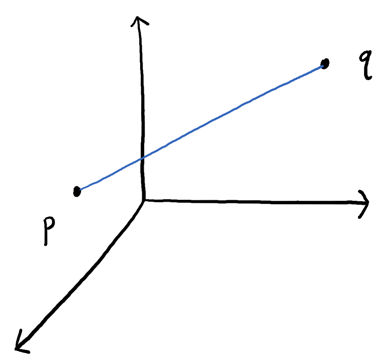


(2) **Straight lines are curves of zero curvature.** Again let $\sigma(t) = p + tN$ be a straight line at p in the direction N. Then σ is a regular curve and so from Chapter 2, we know that it's unit tangent and curvature are, respectively

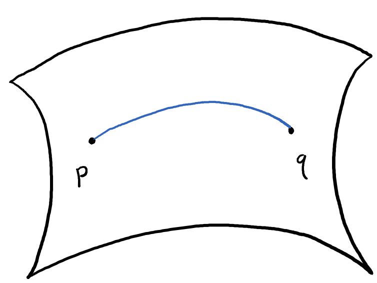
$$T(t) = \frac{\sigma'(t)}{|\sigma'(t)|} = \frac{N}{|N|} \quad \text{and} \quad \kappa(t) = |T'(t)| = 0.$$

Therefore straight lines are curves with zero curvature.

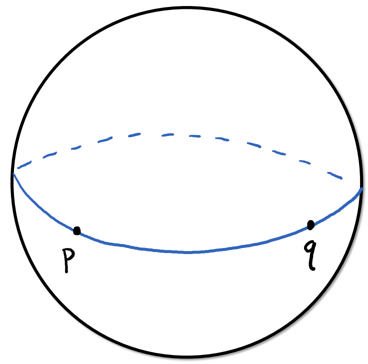
(3) **The shortest distance between two points is a straight line.** Let p and q be two points in $\mathbb{R}^3$. What is the shortest path between p and q? It shouldn't be hard to convince yourself that the shortest path is a straight line between the two points.



The third property readily generalizes to curves on surfaces. We can imagine two points p and q on a surface and ask: what is the shortest path on the surface between p and q?



We can use this idea to motivate the definition of a geodesic. Let's consider the sphere. Consider two points p and q on the sphere.



It shouldn't be hard to convince yourself that the shortest path between the points p and q on a sphere is a great circle between them. By rotating coordinates, we can assume that p and q lie on the equator:

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 0.$$

We can parameterize the equator by

$$\sigma(t) = (\cos t, \sin t, 0).$$

Differentiating σ twice yields

$$\sigma''(t) = (-\cos t, -\sin t, 0).$$

We find $\sigma''(t) = -\sigma(t)$. But $\sigma(t)$ is the radial vector! Hence $\sigma''(t)$ is orthogonal to the sphere S^2. This is the desired definition for a geodesic.

Definition. Let M be a surface. Then a *geodesic* in M is a curve $\sigma(t)$ such that σ'' is orthogonal to M. That is $\sigma''(t) \perp M$ for all t in the domain of σ.

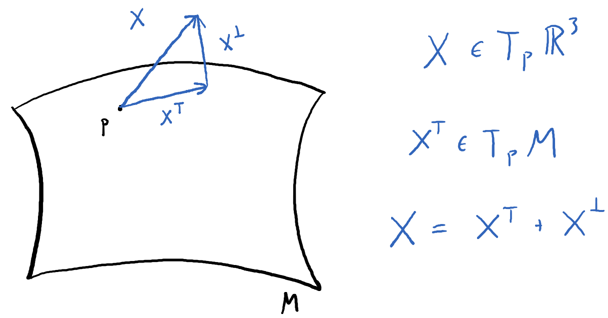
Consider a plane M in $\mathbb{R}^3$ and a straight line $\sigma(t) = p + tN$ in the plane. Since $\sigma''(t) = 0$, we see that we trivially satisfy $\sigma''(t) \perp M$. The fact that $\sigma''(t) = 0$ shows that straight lines in Euclidean space have zero acceleration. We seek a way to generalize this idea to geodesics. However let's recognize that demanding $\sigma'' = 0$ will not work because great circles on spheres have nonzero acceleration but are geodesics!

Covariant Acceleration.

Let M be a surface and consider a tangent vector $X \in T_p\mathbb{R}^3$ at a point $p \in M$. We can decompose X into the sum of two vectors

$$X = X^T + X^\perp$$

where $X^T \in T_pM$ and $X^\perp \in T_p\mathbb{R}^3$ is orthogonal to M. They are called the *tangential* and *perpendicular* components of X, respectively.



We define the operation $\tan : T_p\mathbb{R}^3 \to T_pM$ by

$$\tan(X) = X^T.$$

Now let $\sigma(t)$ be a curve in M. Note that $\sigma''(t) \in T_{\sigma(t)}\mathbb{R}^3$. The *covariant acceleration* of σ is a vector field along σ, denoted by $\frac{D}{dt}\big(\sigma'(t)\big)$, and it's given by

$$\frac{D}{dt}\left(\frac{d\sigma}{dt}\right) = \tan\left(\frac{d^2\sigma}{dt^2}\right).$$

Hence the covariant acceleration of σ is just the tangential component of the ordinary second derivative of σ.

Proposition 6.1. *Let M be a surface. $\sigma(t)$ is a geodesic if and only if its covariant acceleration vanishes; i.e. if and only if $\frac{D}{dt}\sigma'(t) = 0$.*

Proof. Suppose σ is a geodesic. Then $\sigma''(t) \perp M$ for all t. Therefore $\sigma''(t)$ is not tangent to M. Hence $\tan\big(\sigma''(t)\big) = 0$. Hence $\frac{D}{dt}\big(\sigma'(t)\big) = 0$. Conversely, if $\frac{D}{dt}\big(\sigma'(t)\big) = 0$, then $\tan\big(\sigma''(t)\big) = 0$ which implies $\sigma''(t) \perp M$ for all t. $\square$

Because of Proposition 6.1, we call $\frac{D}{dt}\sigma'(t) = 0$ the *geodesic equation*. Solving this equation tells us how to find geodesics in a surface.

Geodesics have constant speeds.

Proposition 6.2. *Let M be a surface and $\sigma(t)$ a geodesic in M. Then $|\sigma'(t)|$ is constant for all t.*

Proof. By definition $|\sigma'(t)| = \sqrt{\langle \sigma'(t), \sigma'(t)\rangle}$. Hence it suffices to show $\langle, \sigma'(t), \sigma'(t)\rangle$ is constant. By the product rule, we have

$$\begin{aligned} \frac{d}{dt}\langle \sigma'(t), \sigma'(t)\rangle &= \langle \sigma''(t), \sigma'(t)\rangle + \langle \sigma'(t), \sigma''(t)\rangle \\ &= 2\langle \sigma''(t), \sigma'(t)\rangle \\ &= 0. \end{aligned}$$

The last fine follows because $\sigma''(t) \perp M$ and $\sigma'(t)$ is tangential to M. Thus $|\sigma'(t)| = c$ for some constant c. $\square$

Let $\sigma(t)$ be a geodesic in a surface M. By the previous proposition, we know that $|\sigma'(t)| = c$ for some constant c. Either $c = 0$ or $c > 0$.

(a) $c = 0$. Then for all t we have $|\sigma'(t)| = 0$ and so $\sigma'(t) = 0$. Hence there exists a $p \in M$ such that $\sigma(t) = p$ for all t. In this case we say that σ is a *trivial geodesic*.

(b) $c > 0$. In this case σ is a regular curve. Therefore we can reparameterize σ in terms of its arc length $s(t) = \int |\sigma'(t)|dt = ct$. We have

$$\frac{d\sigma}{dt} = \frac{d\sigma}{ds}\frac{ds}{dt} = c\frac{d\sigma}{ds}.$$

Since c is a constant, we have

$$\frac{d^2\sigma}{dt^2} = c^2\frac{d^2\sigma}{ds^2}.$$

This gives us the following proposition.

Proposition 6.3. *A unit speed curve $\sigma(s)$ in M is a geodesic if and only if the curuvature vector $T' = \sigma''(s) \perp M$ for all s.*

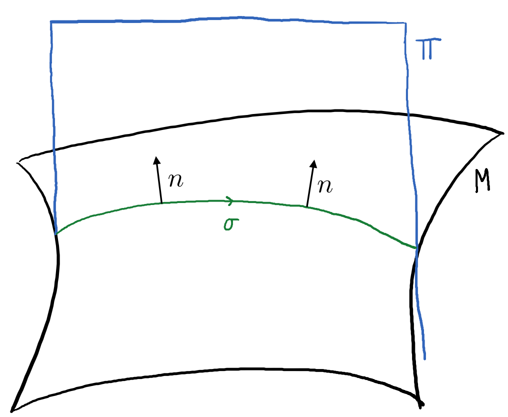
A way to find geodesics.

Proposition 6.4. *Suppose Π is a plane that intersects a surface M orthogonally at every point of intersection. Let $s \to \sigma(s)$ denote the curve of intersection. Then σ is a geodesic.*

Proof. That the plane Π meets M orthogonally at each point means that the at each point of σ the unit normal n to M is tangent to Π. Since σ lies in the plane Π, the unit tangent $T = \sigma'$ is tangent to Π. The curvature vector $T' = \sigma''$ is tangent to Π, as well, as can be seen as follows. Let ν be a unit normal to Π. Then ν is a constant vector field because Π is a plane. Moreover, $\langle \nu, \sigma' \rangle = 0$. Hence,

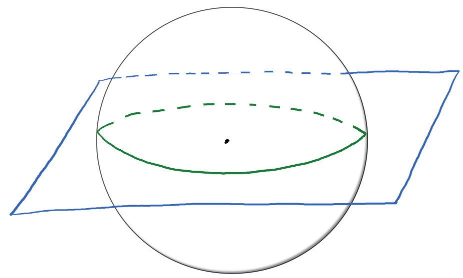
$$0 = \frac{d}{ds}\langle \nu, \sigma' \rangle = \langle \nu', \sigma' \rangle + \langle \nu, \sigma'' \rangle = \langle \nu, \sigma'' \rangle.$$

which implies that $T' = \sigma''$ is tangent to Π. Since T' is tangent to Π and orthogonal to T, it must be proportional to n at each point of σ. It now follows from Proposition 6.3 that σ os a geodesic. $\square$

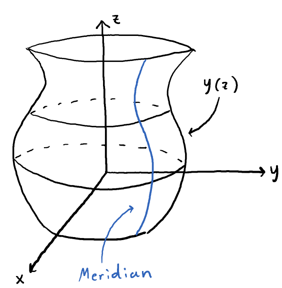


σ = Π ∩ M

Example. Planes that intersect a sphere through the origin produce great circles. From the Proposition 6.4, we know that these great circles are geodesics.



Example. Let's consider a curve $y(z)$ in the $y-z$ plane which is rotated about the z-axis. This will generate a *surface of revolution*. A plane which passes through the origin and is parallel to the z-axis will intersect the surface orthogonally. Therefore the intersection will be a geodesic. These geodesics are called *meridians*.



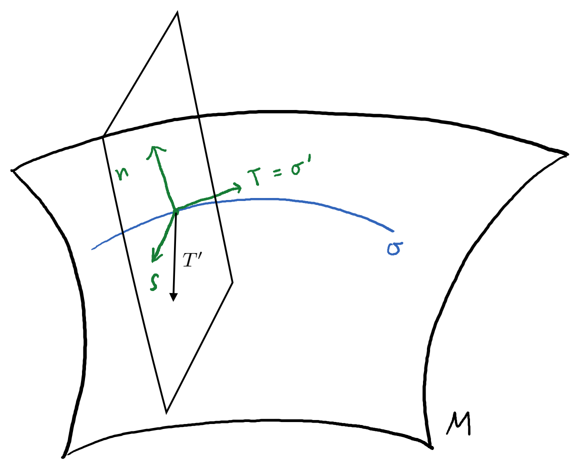
Geodesics and Curvature.

Our goal in this section is to understand the relationship between geodesics and the curvature of curves we discussed in chapter 2.

Let $\sigma(s)$ be a regular curve on a surface M with a unit speed parameterization. Recall that the *unit tangent* to σ is the vector field $T(s)$ along σ defined by

$$T(s) = \sigma'(s).$$

Let n be a unit normal to M. When n is restricted to σ, we have a vector field $n(s)$ along σ. We define the vector field $S(s) = T(s) \times n(s)$ along σ.



$\{S, T, n\}$ is an orthonormal basis of vectors along σ. Therefore given any $X \in T_p\mathbb{R}^3$ for some p on the image of σ, we can decompose X in terms of the basis vectors:

$$X = \langle X, T\rangle T + \langle X, S\rangle S + \langle X, n\rangle n$$

Consider $X = \sigma''$, then decomposing σ'' gives

$$\sigma'' = \langle \sigma'', T\rangle T + \langle \sigma'', S\rangle S + \langle \sigma'', n\rangle n$$

Then since $\langle \sigma'', T\rangle = 0$ we have,

$$\sigma'' = \langle \sigma'', S\rangle S + \langle \sigma'', n\rangle n$$

We define the *geodesic curvature* as $\kappa_g = \langle \sigma'', S\rangle$ and recall that $\kappa_n = \langle \sigma'', n\rangle$ is the normal curvature of σ. Thus, the curvature vector may be expressed as,

$$\sigma'' = \kappa_g S + \kappa_n n\,.$$

Proposition 6.5. *Let σ be a regular curve in M. Then σ is a geodesic if and only if its geodesic curvature vanishes (i.e. $\kappa_g = 0$).*

Proof. Since σ is a regular curve, we can write its second derivative as $\sigma'' = \kappa_g S + \kappa_n n$. If σ is a geodesic, then $\sigma''(s) \perp M$ for all s. That is σ'' is proportional to n, and so $\kappa_g = 0$. Conversely, if $\kappa_g = 0$, then σ'' is proportional to n. $\square$

Existence of Geodesics.

In this section we will establish the existence of geodesics in a surface. This will be accomplished by the fundamental existence and uniqueness theorem of ordinary differential equations. In order to use this theorem, we need to express the geodesic equation, $\frac{D}{dt}\frac{d\sigma}{dt} = 0$, in a more tangible form with coordinates.

Let $\sigma(t)$ be a curve in a surface M. Recall that the covariant acceleration of σ is given by

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right)$$

and Proposition 6.1 showed us that σ is a geodesic if and only if $\frac{D}{Dt}\sigma' = 0$. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ be a proper coordinate patch. The coordinates for U will be u^1 and u^2. In terms of these coordinates, we can write a curve $\sigma(t)$ as

$$\sigma(t) = \mathbf{x}\big(u^1(t), u^2(t)\big).$$

Then in terms of these coordinates, we have

$$\begin{aligned} \frac{d^2\sigma}{dt^2} &= \frac{d}{dt}\left(\frac{d\sigma}{dt}\right) \\ &= \frac{d}{dt}\left(\frac{d}{dt}\mathbf{x}\big(u^1(t), u^2(t)\big)\right) \\ &= \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{du^2}{dt}\right). \end{aligned}$$

Recall that we use the notation $\mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u^i}$ for $i = 1, 2$. Therefore

$$\begin{aligned} \frac{d^2\sigma}{dt^2} &= \frac{d}{dt}\left(\sum_{i=1}^{2}\frac{du^i}{dt}\mathbf{x}_i\right) \\ &= \sum_{i=1}^{2}\left(\frac{d^2u^i}{dt^2}\mathbf{x}_i + \frac{du^i}{dt}\frac{d\mathbf{x}_i}{dt}\right). \end{aligned}$$

Plugging in

$$\frac{d\mathbf{x}_i}{dt} = \frac{\partial \mathbf{x}_i}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \mathbf{x}_i}{\partial u^2}\frac{du^2}{dt} = \mathbf{x}_{i1}\frac{du^1}{dt} + \mathbf{x}_{i2}\frac{du^2}{dt}$$

we get

$$\frac{d^2\sigma}{dt^2} = \sum_{i=1}^{2}\left(\frac{d^2u^i}{dt^2}\mathbf{x}_i + \frac{du^i}{dt}\sum_{j=1}^{2}\frac{du^j}{dt}\mathbf{x}_{ij}\right).$$

Recall from the previous chapter that we derived the following expression for $\mathbf{x}_{ij}$:

$$\mathbf{x}_{ij} = \sum_{k=1}^{2}\Gamma_{ij}^{k}\mathbf{x}_k + L_{ij}n$$

where n is a unit normal to M. Therefore

$$\frac{d^2\sigma}{dt^2} = \sum_{i=1}^{2}\left(\frac{d^2u^i}{dt^2}\mathbf{x}_i + \frac{du^i}{dt}\sum_{j=1}^{2}\frac{du^j}{dt}\left[\sum_{k=1}^{2}\Gamma_{ij}^{k}\mathbf{x}_k + L_{ij}n\right]\right).$$

Note in the above equation, the only part that is not tangent to M is $L_{ij}n$. Therefore the covariant acceleration of σ is:

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right) = \sum_{i=1}^{2}\frac{d^2u^i}{dt^2}\mathbf{x}_i + \sum_{i,j,k=1}^{2}\frac{du^i}{dt}\frac{du^j}{dt}\Gamma_{ij}^{k}\mathbf{x}_k.$$

By relabeling the index 'i' to a 'k' in the first term, we obtain

$$\frac{D}{dt}\frac{d\sigma}{dt} = \tan\left(\frac{d^2\sigma}{dt^2}\right) = \sum_{k=1}^{2}\left(\frac{d^2u^k}{dt^2} + \sum_{i,j=1}^{2}\frac{du^i}{dt}\frac{du^j}{dt}\Gamma_{ij}^{k}\right)\mathbf{x}_k.$$

Thus the geodesic equation, $\frac{D}{dt}(\sigma') = 0$, is equivalent to

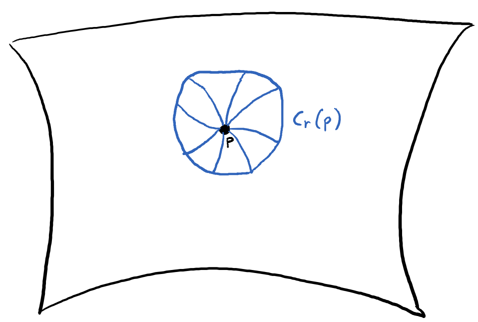
$$\frac{d^2u^k}{dt^2} + \sum_{i,j=1}^{2}\Gamma_{ij}^{k}\frac{du^i}{dt}\frac{du^j}{dt} = 0 \quad \text{for each} \quad \text{k} = 1, 2.$$

This is a system of differential equations in the four unknowns $\frac{du^1}{dt}$, $\frac{du^2}{dt}$, $\frac{d^2u^1}{dt^2}$, and $\frac{d^2u^2}{dt^2}$. Since each Γ_{ij}^{k} is a smooth function on the coordinate patch, the fundamental existence and uniqueness theorem of ordinary differential equations says that given initial conditions on u^1, u^2, $\frac{du^1}{dt}$, and $\frac{du^2}{dt}$, the unknowns can be solved and the solution is unique for some maximal time interval. Thus we have the following theorem.

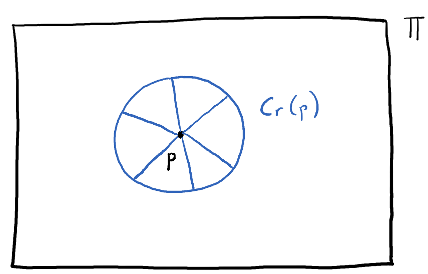
Theorem 6.6. *Given any $p \in M$ and any $X \in T_pM$, there exists a geodesic $\sigma(t)$ satisfying $\sigma(0) = p$ and $\sigma'(0) = X$. There is a maximal time interval on which the geodesic is unique.*

The vector $X \in T_pM$ in the above theorem was arbitrary. By allowing X to vary, we can consider geodesics emanating from p with various initial velocities. Given a radius $r > 0$, we define $C_r(p)$, called the *geodesic circle* of radius r centered at p, to be the set of points $q \in M$ such that there is a geodesic σ starting at p and ending at q which has length r. The following proposition is intuitively believable, but deceptively difficult to prove. It's normally proved in more advance differential geometry courses.

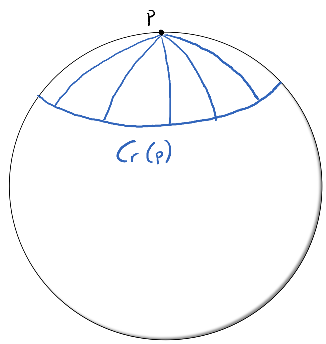
Proposition 6.7. *Given any $p \in M$ there exists a (small) radius $r > 0$ such that $C_r(p)$ can be parameterized by a smooth closed curve.*



Example. For a plane Π in $\mathbb{R}^3$, geodesic circles are just usual circles.



Example. Consider the sphere S^2 and a geodesic circle $C_r(p)$ centered about $p \in S^2$. By rotating coordinates, we can assume that p is at the north pole. Then $C_r(p)$ corresponds to a circle of constant latitude.

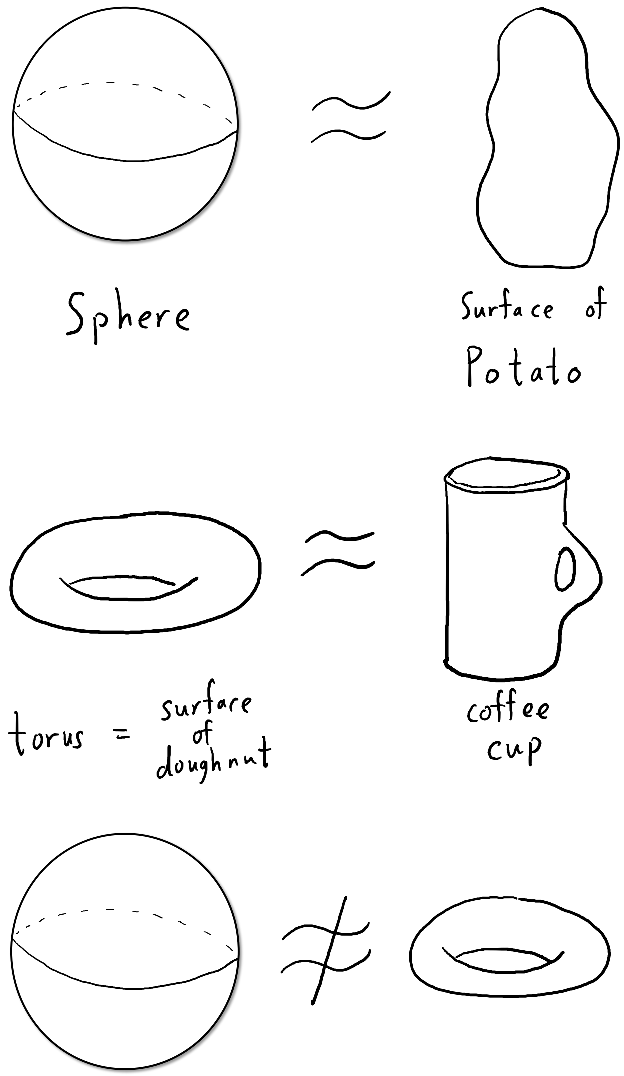


Gauss-Bonnet Theorem.

The Gauss-Bonnet Theorem is a beautiful connection between the geometry and topology of surfaces. The geometry part has to deal with curvature which we have discussed in great detail already. So let's acquaint ourselves with the topology part. This field of study goes under the name *algebraic topology* and is normally taught in graduate courses. For our purposes we only want to understand the bare minimum, so we can quickly see how curvature plays a role via the Gauss-Bonnet theorem. Therefore we will not prove every theorem about topology stated in this section, but we will use lots of examples to portray the theorems.

Topology of Surfaces

We will say two surfaces M and N are *topologically equivalent* or are of the same *topological type* if M and N are diffeomorphic. We normally write $M \approx N$ to denote this. For example the surface of a sphere, potato, and dumbell are all of the same topological type because they are all diffeomorphic to each other. The surface of a doughnut is topologically equivalent to that of a coffee cup. To see how these surfaces are diffeomorphic to each other, one can imagine that each of these surfaces is made of play-doh. We can mold and shift the play-doh of some surface to turn it into another surface. However it's impossible to mold the surface of a sphere into that of a torus without puncturing a hole in the surface. This ‘puncturing’ creates a discontinuity in any type of mapping one can create from the sphere to the torus, and so they cannot be diffeomorphic.

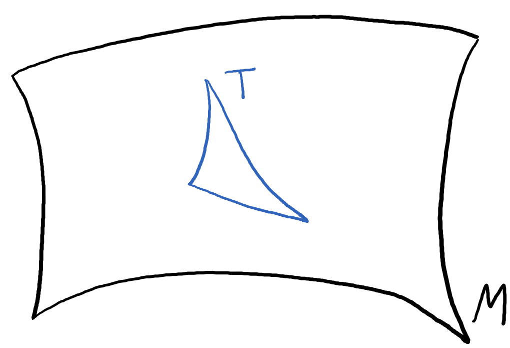


Of course using play-doh to describe surfaces is not mathematically rigorous, but these ideas can be made rigorous which is what courses on algebraic topology set out to do.

Euler Number

It's an astonishing fact that to determine whether two surfaces are topologically equivalent (diffeomorphic) comes down to computing exactly one number of that surface. That number is called the *Euler characteristic*. If both surfaces have the same number, then they have the same topological type. Before defining the Euler characteristic, we first have to talk about how to triangulate surfaces.

Definition. A *triangle* T in M is a simple region in M bounded by 3 smooth curve segments. (Here 'simple' means that T is topologically a disk.) The curves which make up the triangle are called *edges* and the points where one curve ends and another begins are called *vertices*. The region bounded by the edges is called the *face*. A *geodesic triangle* is a triangle T whose edges are geodesics. We will almost exclusively be working with geodesic triangles.



Definition. A *triangulation* of M is a decomposition of M into a finite number of triangles $T_1, T_2, \ldots, T_n$ such that

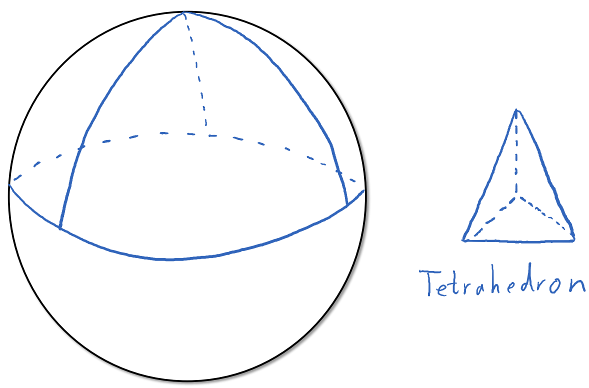
(1) $\bigcup_{i=1}^{n} T_i = M$

(2) If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common edge or a vertex.

It's a fact (we will not prove) that every compact surface can be triangulated.

Theorem 6.8. *Every compact surface admits a triangulation. Moreover we can assume the edges of the triangulation are geodesics.*

Example. The figure below shows a triangulation of the sphere. Note that the edges of the triangles are great circles and hence geodesics. The triangulation has the same topology type as a tetrahedron. The number of faces is $F = 4$, the number of edges is $E = 6$, and the number of vertices is $V = 4$.



Definition. Let M be a compact surface and consider any triangulation of M. Then the *Euler characteristic* of M is

$$\chi(M) = V - E + F$$

where

$$\begin{aligned} V &= \text{number of vertices} \\ E &= \text{number of edges} \\ F &= \text{number of faces} \end{aligned}$$

The following fact (we will not prove) justifies the definition of $\chi(M)$.

Theorem 6.9. *The Euler characteristic $\chi(M)$ does not depend on the particular triangulation of M.*

It can be shown (in more advance courses) that $\chi(M)$ is a topological invariant. That is, it does not depend on the topological type of the surface. This gives us the following theorem.

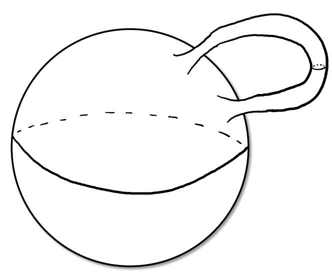
Theorem 6.10. *Two surfaces M and N are topologically equivalent if and only if $\chi(M) = \chi(N)$. In other words*

$$M \approx N \iff \chi(M) = \chi(N).$$

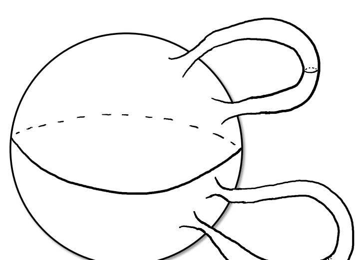
Example. Find a triangulation of the torus T^2 and convince yourself that $\chi(T^2) = 0$. Since the Euler characteristic of the sphere is $\chi(S^2) = 2$, the above theorem tells us that the torus and the sphere are not diffeomorphic (which is what we expect).

Genus and classification of compact surfaces in $\mathbb{R}^3$.

There is an easy way to construct surfaces with different topology. The idea is to 'glue' handles onto a sphere.



Sphere with one handle attached = genus one surface



Sphere with two handles attached = genus two surface

Definition. When we construct a surface M in this way with g handles, then we say M is a *surface of genus g*.

There is a simple relationship between the genus and the Euler characteristic.

Proposition 6.11. *If M is a surface of genus g, then $\chi(M) = 2(1-g)$.*

The proof follows from a formula involving the *connected sum* of two surfaces: $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

Example. The sphere S^2 is a surface of genus zero. Therefore $\chi(S^2) = 2$. This agrees with our triangulation of the sphere. The torus T^2 is a surface of genus one. Therefore $\chi(T^2) = 0$. Make sure this agrees with the triangulation you found of T^2 in the previous example.

One of the triumphs of elementary differential geometry is the following classification theorem.

Theorem 6.12 (Classification of compact surfaces)**.** *Every compact surface in $\mathbb{R}^3$ is diffeomorphic to a genus g surface.*

Theorem 6.13 (Gauss-Bonnet Theorem)**.** *Let M be a compact surface in $\mathbb{R}^3$ with Gaussian curvature K. Then*

$$\iint_M K dA = 2\pi\chi(M)$$

Thus the Gauss-Bonnet theorem gives us a relationship between topology and the average curvature of a surface. Here is an interesting consequence of these two theorems.

Corollary 6.14. *Suppose M has positive Gaussian curvature everywhere. Then M is diffeomorphic to a sphere.*

Proof. By assumption we have $K > 0$ on the sphere which implies $\chi(M) > 0$. By Proposition 6.11, we know that $\chi(M) = 2(1-g)$. Since g can only take on the values $g = 0, 1, 2, \ldots$, we must have $g = 0$. By the classification theorem of compact surfaces, it follows that M is topologically a sphere. $\square$

Our main tool we will use to prove the Gauss-Bonnet theorem is the Angle Excess theorem.

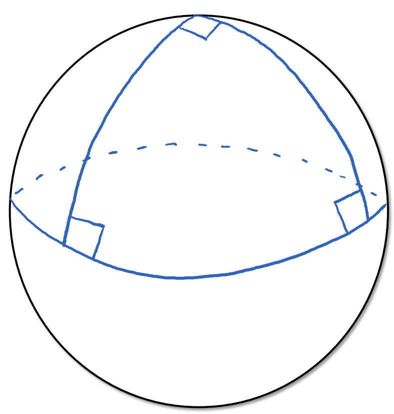
Theorem 6.15 (Angle Excess Theorem)**.** *Let T be a geodesic triangle with angles A, B, and C. Then*

$$A + B + C = \pi + \iint_T K dS.$$

We will prove the angle excess theorem after we use it to prove the Gauss-Bonnet theorem. For now we will illustrate the angle excess theorem with some examples.

Example. Let $M = \mathbb{R}^2$. Then $K = 0$ and we have the usual theorem from Euclidean geometry: $A + B + C = \pi$.

Example. Let $M = S_R^2$ be the sphere with radius R. Using great circles (i.e. geodesics), we will draw three arcs which connect at 90 degree angles. See figure

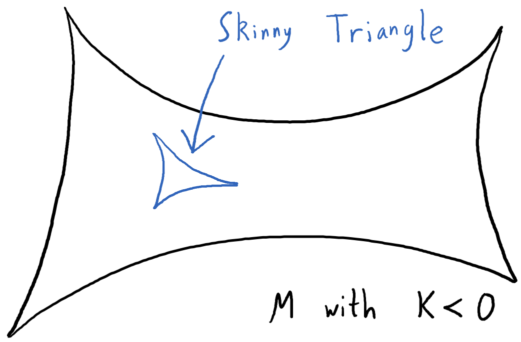


Then we should have $A + B + C = \frac{3}{2}\pi$. Now let's use the angle excess theorem to confirm this. We have

$$\iint_T K dS = \iint_T \frac{1}{R^2} dA = \frac{1}{R^2} \iint_T dA = \frac{1}{R^2}\left(\frac{4\pi R^2}{8}\right) = \frac{\pi}{2}.$$

Thus the angle excess theorem agrees with our picture.

From the angle excess theorem, we see that if $K > 0$, then $A+B+C > \pi$ which gives us *fat* triangles. If $K < 0$, then $A + B + C < \pi$ which gives us *skinny* triangles.



Proof of the Gauss-Bonnet Theorem. By Theorem 6.8, we can triangulate M with geodesic edges. Let $T_1, T_2, \ldots, T_n$ be the triangles in the triangulation. Then note that the number of faces is $F = n$. For each $i = 1, 2, \ldots, n$, let A_i, B_i, and C_i be the angles for each geodesic triangle T_i. Then the Angle Excess Theorem gives us

$$\iint_M K dS = \sum_{i=1}^{n} \iint_{T_i} K dS = \sum_{i=1}^{n} (A_i + B_i + C_i - \pi) = \sum_{i=1}^{n} (A_i + B_i + C_i) - n\pi.$$

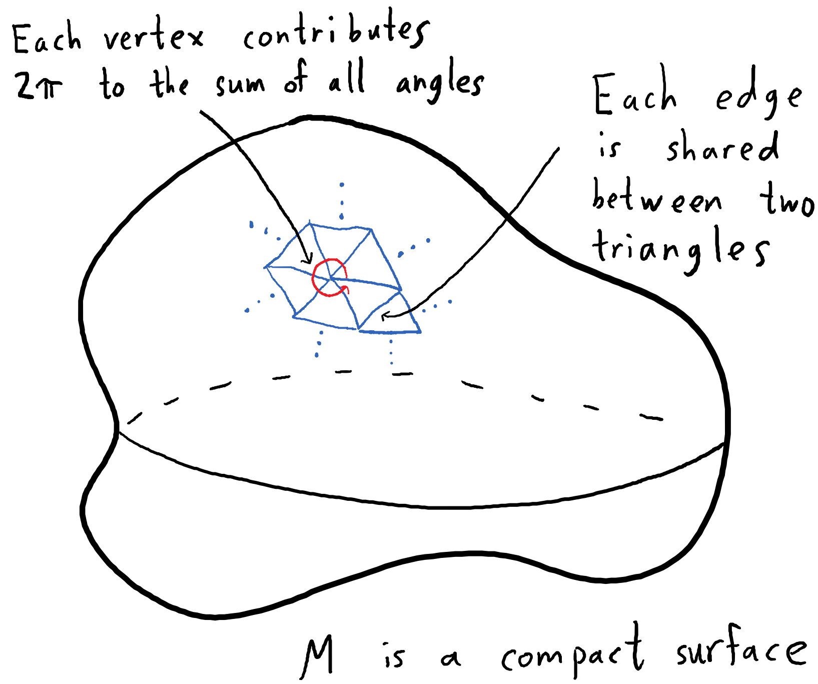
Notice that since the triangles must cover M entirely, the sum of all the angles put together must equal 2π times the number of vertices V. See the figure below. Therefore $\sum_{i=1}^{n}(A_i + B_i + C_i) = 2\pi V$, and so we have

$$\iint_M K dS = 2\pi V - n\pi = 2\pi V - \pi F$$

Now let's count how many edges there have to be. We might expect that for every face, there are three edges: $E = 3F$. However recognize that we are double counting the edges, because each edge shares the face of two triangles. See the figure below. Therefore the number of edges is really $E = \frac{3}{2}F$. Thus

$$\iint_M K dS = 2\pi V - \pi F = 2\pi \left(V + F - \frac{3}{2}F \right) = 2\pi(V + F - E) = 2\pi\chi(M).$$

□

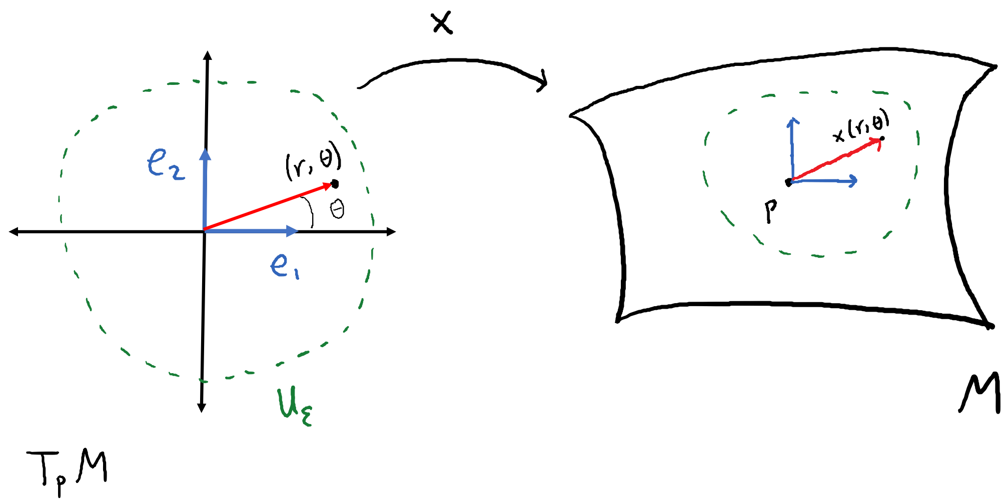


Proof of the Angle Excess Theorem.

In this section we will prove the angle excess theorem. The techniques involved in this proof also appear in graduate differential geometry courses, so if you understand this proof, then you will be well prepared for the more advanced stuff.

Geodesic Polar Coordinates

First we need to introduce a good choice of coordinates made out of geodesics. Consider a point p in a surface M and its tangent space T_pM. T_pM is a two-dimensional vector space based at p. Let $\{e_1, e_2\}$ be an orthonormal basis for T_pM consider an angle θ traveling counterclock wise so that the right hand rule points in the direction normal to the surface. See the figure below. For $\epsilon > 0$ let $U_\epsilon = \{(r, \theta) \mid 0 < r < \epsilon, 0 < \theta < 2\pi\}$ and define $\mathbf{x} : U_\epsilon \to M$ by $\mathbf{x}(r, \theta) =$ point in M reached by traveling along the geodesic for a length of r starting at p and in the direction θ.



For reasons related to Lie group theory, $\mathbf{x}$ is known as the *exponential map*. r and θ are called *geodesic polar coordinates*. Similar to Proposition 6.7, we have the following

Proposition 6.16. *For ϵ sufficiently small, $\mathbf{x} : U_\epsilon \to M$ is a proper coordinate patch.*

We will always assume that $\epsilon > 0$ is small enough so that $\mathbf{x} : U_\epsilon \to M$ is a proper coordinate patch. In such a patch, the curves $r \mapsto (r, \theta_0)$ for a constant θ_0 are called *radial geodesics* emanating from p. The curves $\theta \mapsto (r_0, \theta)$ for a constant r_0 are called *geodesic circles* based at p.

Consider the components of the induced metric in this coordinate patch. Since are coordinates are r and θ, our components will be g_{rr}, $g_{r\theta}$, and $g_{\theta\theta}$. g_{rr} is easy to calculate. Recall $\mathbf{x}_r = \frac{\partial \mathbf{x}}{\partial r}$. Then

$$g_{rr} = \langle \mathbf{x}_r, \mathbf{x}_r \rangle = |\mathbf{x}_r|^2 = 1. \tag{6.1}$$

The last equality follows from the fact that the coordinate r measures arc-length. This is from the definition of the exponential map $\mathbf{x}$. The following explains the utility of geodesic polar coordinates.

Lemma 6.17 (Gauss Lemma)**.** *In geodesic polar coordinates, we always have $g_{r\theta} = 0$, i.e. they form an orthogonal coordinate system.*

Proof. Consider a point $p \in M$ and the exponential map $\mathbf{x} : U_\epsilon \to M$ based at p. Let $q \in U_\epsilon$. We want to show $g_{r\theta}(q) = \langle \mathbf{x}_r, \mathbf{x}_\theta \rangle(q) = 0$. Let $\sigma(r)$ be the unit speed geodesic from p to q. By definition of the exponential map $\mathbf{x}_{rr}$ corresponds to σ''. Smoothness of the exponential map implies that $\mathbf{x}_\theta(p) = 0$. Therefore $\langle \mathbf{x}_r, \mathbf{x}_\theta \rangle(p) = 0$. Thus it suffices to show that $\langle \mathbf{x}_r, \mathbf{x}_\theta \rangle \big(\sigma(r)\big)$ is independent of r.

Since $\sigma'' \perp M$ by definition of geodesics, we have $\langle \mathbf{x}_{rr}, \mathbf{x}_\theta \rangle = 0$. So by the product rule, we obtain the following

$$\frac{\partial}{\partial r} \langle \mathbf{x}_r, \mathbf{x}_\theta \rangle = \langle \mathbf{x}_{rr}, \mathbf{x}_\theta \rangle + \langle \mathbf{x}_r, \mathbf{x}_{\theta r} \rangle = 0 + \langle \mathbf{x}_r, \mathbf{x}_{\theta r} \rangle.$$

Combining this with

$$\frac{\partial}{\partial \theta} |\mathbf{x}_r|^2 = \frac{\partial}{\partial \theta} \langle \mathbf{x}_r, \mathbf{x}_r \rangle = \langle \mathbf{x}_{r\theta}, \mathbf{x}_r \rangle + \langle \mathbf{x}_r, \mathbf{x}_{r\theta} \rangle = 2\langle \mathbf{x}_r, \mathbf{x}_{r\theta} \rangle,$$

we have

$$\frac{\partial}{\partial r} \langle \mathbf{x}_r, \mathbf{x}_\theta \rangle = \frac{1}{2} \frac{\partial}{\partial \theta} |\mathbf{x}_r|^2 = 0.$$

The last equality follows from Proposition 6.2. □

<u>Gaussian Curvature in Geodesic Polar Coordinates</u>

Consider a point p in a surface M and introduce geodesic polar coordinates (r, θ) around p via the exponential map $\mathbf{x} : U_\epsilon \to M$. We want to compute the Gaussian curvature K of M in terms of these coordinates. First some notation.

$$ds^2 = \sum_{i,j} g_{ij} dx^i dx^j = g_{rr} dr^2 + 2g_{r\theta} dr d\theta + g_{\theta\theta} d\theta^2.$$

The Gauss Lemma tells us that $g_{r\theta} = 0$. Also, from Equation 6.1 we know that $g_{rr} = 1$. Therefore, if we let $f = \sqrt{g_{\theta\theta}}$, then the metric in geodesic polar coordinates is simply

$$ds^2 = dr^2 + f^2 d\theta^2. \tag{6.2}$$

Since geodesic polar coordinates are intrinsically defined (i.e. we did not use the coordinates from $\mathbb{R}^3$ to define them), we should calculate the Gaussian curvature via intrinsic quantities. Recall Gauss' Theorem Egregium

(Theorem 5.17) says that the Gaussian curvature can be computed in terms of the g_{ij}'s. In the proof of that theorem we found that

$$K = \sum_{l=1}^{2} \frac{g_{2l} R^l_{121}}{g} \tag{6.3}$$

In geodesic polar coordinates, we know from equation (6.2) that the components g_{ij} are simply

$$g_{rr} = 1, \quad g_{r\theta} = 0, \quad g_{\theta\theta} = f^2\,. \tag{6.4}$$

SInce this is an orthogonal coordinate system, one can apply Equation 5.26 to obtain (exercise!)

$$K = -\frac{1}{f}\frac{\partial^2 f}{\partial r^2}. \tag{6.5}$$

Finally we are able to prove:

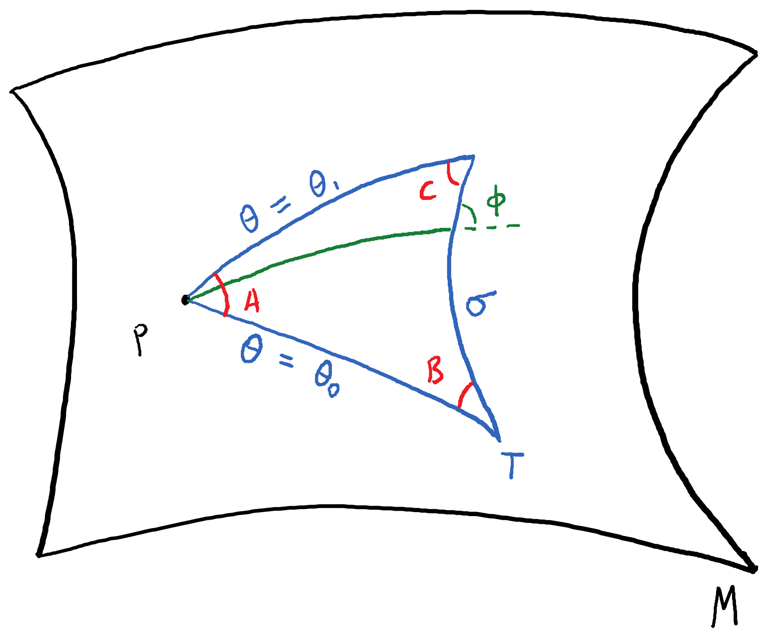
Theorem 6.18 (Angle Excess Theorem)**.** *Let T be a geodesic triangle with angles A, B, and C. Then*

$$A + B + C = \pi + \iint_T K dS.$$

Proof. We will break the proof up into three steps.

Step 1: Set up

Let's construct a geodesic triangle T with vertex p in a surface S with angles A, B, and C. We use geodesic polar coordinates (r, θ) to parameterize T. Let's say the two edges of T emanating from p correspond to values $\theta = \theta_0$ and $\theta = \theta_1$. The other edge we will call σ. That is, σ is the edge opposite that of p. In the figure below, the green curve is a radial geodesic emanating from p with angle θ where $\theta_0 \leq \theta \leq \theta_1$. ϕ is the angle between these radial geodesics and σ. Therefore ϕ is a function θ.



Step 2: Showing $\frac{d\phi}{d\theta} = -\frac{\partial f}{\partial r}$

We parameterize σ with respect to arclength,

$$\sigma(s) = \mathbf{x}\big(r(s), \theta(s)\big). \tag{6.6}$$

We use $' = \frac{d}{ds}$ to denote differentiation with respect to arclength. Then $\sigma' = r'\mathbf{x}_r + \theta'\mathbf{x}_\theta$. So since the parameterization is with respect to arclength and our metric is given by $ds^2 = dr^2 + f^2 d\theta^2$, we have

$$1 = \langle \sigma', \sigma' \rangle = \langle r'\mathbf{x}_r + \theta'\mathbf{x}_\theta, r'\mathbf{x}_r + \theta'\mathbf{x}_\theta \rangle = (r')^2 + (\theta')^2 f^2. \tag{6.7}$$

Since σ' and $\mathbf{x}_r$ are unit vectors, we have $\langle \sigma', \mathbf{x}_r \rangle = \cos\phi$. Therefore

$$\cos\phi = \langle r'\mathbf{x}_r + \theta'\mathbf{x}_\theta, \mathbf{x}_r \rangle = r'\langle \mathbf{x}_r, \mathbf{x}_r \rangle + \theta'\langle \mathbf{x}_\theta, \mathbf{x}_r \rangle = r'. \tag{6.8}$$

The last equality follows from the Gauss Lemma (Lemma 6.17). Plugging equation (6.8) into equation (6.7) yields

$$1 = \cos^2\phi + (\theta')^2 f^2 \tag{6.9}$$

Hence

$$\sin\phi = \theta' f. \tag{6.10}$$

Therefore we have an expression for $\theta' = d\theta/ds$, so in order to find $d\theta/d\phi$ all we need is an expression for $\phi' = d\phi/ds$. Differentiating $\cos\phi = \langle\sigma', \mathbf{x}_r\rangle$, we get

$$-\phi'\sin\phi = \frac{d}{ds}\langle\sigma', \mathbf{x}_r\rangle = \langle\sigma'', \mathbf{x}_r\rangle + \langle\sigma', \mathbf{x}_r'\rangle = \langle\sigma', \mathbf{x}_r'\rangle. \tag{6.11}$$

The second equality follows because σ is a geodesic. An application of the chain rule gives

$$-\phi'\sin\phi = \langle\sigma', \mathbf{x}_{rr}r' + \mathbf{x}_{r\theta}\theta'\rangle = \langle\sigma', \mathbf{x}_{r\theta}\theta'\rangle = \theta'\langle\sigma', \mathbf{x}_{r\theta}\rangle. \tag{6.12}$$

The second equality follows since $\mathbf{x}_{rr} = 0$ because we're using geodesic polar coordinates. This gives us the following expression for $d\theta/d\phi$

$$\frac{d\phi}{d\theta} = \frac{\phi'}{\theta'} = -\frac{\langle\sigma', \mathbf{x}_{r\theta}\rangle}{\sin\phi} = -\frac{\langle\sigma', \mathbf{x}_{r\theta}\rangle}{\theta' f} \tag{6.13}$$

We used equation (6.10) in the last equality. Now we want to simplify $\langle\sigma', \mathbf{x}_{r\theta}\rangle$. We have

$$\langle\sigma', \mathbf{x}_{r\theta}\rangle = \langle r'\mathbf{x}_r + \theta'\mathbf{x}_\theta, \mathbf{x}_{r\theta}\rangle = r'\langle\mathbf{x}_r, \mathbf{x}_{r\theta}\rangle + \theta'\langle\mathbf{x}_\theta, \mathbf{x}_{r\theta}\rangle. \tag{6.14}$$

Let's evaluate these two terms separately.

$$\langle\mathbf{x}_r, \mathbf{x}_{r\theta}\rangle = \frac{1}{2}\frac{\partial}{\partial\theta}\langle\mathbf{x}_r, \mathbf{x}_r\rangle = \frac{1}{2}\frac{\partial}{\partial\theta}(1) = 0. \tag{6.15}$$

$$\langle\mathbf{x}_\theta, \mathbf{x}_{r\theta}\rangle = \frac{1}{2}\frac{\partial}{\partial r}\langle\mathbf{x}_\theta, \mathbf{x}_\theta\rangle = \frac{1}{2}\frac{\partial}{\partial r}(f^2) = f\frac{\partial f}{\partial r} \tag{6.16}$$

Plugging these into equation (6.14), we get

$$\langle\sigma', \mathbf{x}_{r\theta}\rangle = \theta' f\frac{\partial f}{\partial r}. \tag{6.17}$$

Finally, plugging this into equation (6.13), we obtain

$$\frac{d\phi}{d\theta} = -\frac{\partial f}{\partial r}. \tag{6.18}$$

This completes step 2.

Step 3: Evaluating $\iint_T K dS$

Let's put all the pieces together. In the section before this, we calculated the Gaussian curvature in geodesic polar coordinates $K = -\frac{1}{f}\frac{\partial^2 f}{\partial r^2}$. The area element in geodesic polar coordinates is $dS = \sqrt{g} dr d\theta = f dr d\theta$ since $g = \det[g_{ij}] = f^2$. Therefore

$$\iint_T K dS = \iint_T \left(-\frac{1}{f}\frac{\partial^2 f}{\partial r^2}\right) f dr d\theta = -\iint_T \frac{\partial^2 f}{\partial r^2} dr d\theta. \tag{6.19}$$

Putting bounds on our integral, we get

$$\iint_T K dS = -\int_{\theta_0}^{\theta_1}\int_0^{r(\theta)} \frac{\partial^2 f}{\partial r^2} dr d\theta = -\int_{\theta_0}^{\theta_1}\left(\frac{\partial f}{\partial r} - 1\right) d\theta. \tag{6.20}$$

Note that we used $\frac{\partial f}{\partial r}\Big|_{r=0} = 1$ (convince yourself why this is true). Rearranging and using $d\phi/d\theta = -\partial f/\partial r$ gives

$$\iint_T K dS = \int_{\theta_0}^{\theta_1} d\theta - \int_{\theta_0}^{\theta_1}\frac{\partial f}{\partial r} d\theta = A + \int_{\theta_0}^{\theta_1}\frac{d\phi}{d\theta} d\theta. \tag{6.21}$$

From the figure we see that

$$\iint_T K dS = A + \int_{\phi(\theta_0)}^{\phi(\theta_1)} d\phi = A + \phi(\theta_1) - \phi(\theta_0) = A + C - (\pi - B) \tag{6.22}$$

Thus

$$\iint_T K dS = A + B + C - \pi. \tag{6.23}$$

$\square$

Chapter 6 Exercises

1. Let $s \to \sigma(s)$ be a unit speed curve which lies on a sphere of radius R.

   (a) Show that the normal curvature κ_n of σ (with respect to the outward normal) is equal to $-\frac{1}{R}$. (Hint: Use the geometric interpretation of the second fundamental form $\mathcal{L}$.)

   (b) Using the result of Exercise 2.7 on p. 32, show that if σ has constant geodesic curvature, $\kappa_g =$ const., then σ is a circle (or part of a circle).

   (c) Show that any geodesic on a sphere is a great circle (or part of a great circle). Hint: Use part (b).

2. Let M be the surface of revolution obtained by rotating the curve $\sigma(t) = (r(t), 0, z(t))$ about the z-axis, as decribed on pp 52-53. Assume that σ is a *unit speed* curve.

   (a) Show that, with respect to the coordinates (t, θ), the metric components are given by: $g_{tt} = 1$, $g_{t\theta} = 0$, and $g_{\theta\theta} = r^2$.

   (b) Consider the curve $\tau \to \gamma(\tau)$ expressed in terms of coordinates as, $\gamma(\tau) = \mathbf{x}(t(\tau), \theta(\tau))$. Compute the Christoffel symbols Γ^k_{ij}, i.e., Γ^t_{tt}, $\Gamma^t_{t\theta} = \Gamma^t_{\theta t}$, etc. to obtain the geodesic equations,

   $$t'' - r\dot{r}(\theta')^2 = 0 \tag{6.24}$$

   $$\theta'' + 2\frac{\dot{r}}{r}t'\theta' = 0\,, \tag{6.25}$$

   where $' = d/d\tau$ and $\cdot = d/dt$.

   (c) Suppose γ is a unit speed geodesic. Show by differentiation, and the second geodesic equation, that the quantity $r^2\theta'$ is constant along γ. Show that this implies $\langle \gamma', \mathbf{x}_\theta \rangle =$ const along γ. (Remark: Using that γ' is a unit vector, this, in turn, implies that $r\cos\phi =$ const along γ, where ϕ is the angle between γ and the latitudinal circles. This is known as *Clairaut's relation*).

3. Let p be the north pole of the sphere of radius R, S^2_R, and let $C_r(p)$ be the geodesic circle of radius r ($r < R$) centered at p.

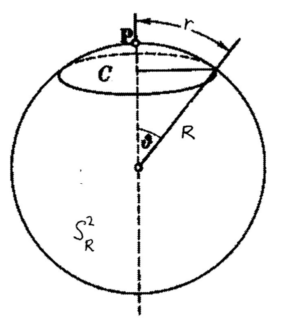
   (a) Show by elementary geometry (see the figure) that $C_r(p)$ is the circle of latitude at co-latitude $\theta = \frac{r}{R}$. Show that the length of $C_r(p)$ is given by,

   $$L(C_r(p)) = 2\pi R \sin\left(\frac{r}{R}\right) .$$

   (b) Using the Maclaurin series of the sine to expand $\sin\left(\frac{r}{R}\right)$ in powers of $\frac{r}{R}$, show that,

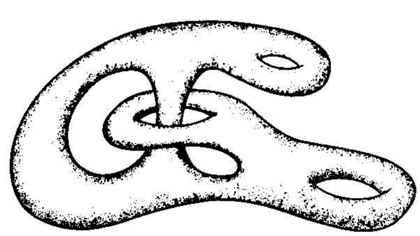
   $$L(C_r(p)) = 2\pi r - \frac{\pi}{3}\frac{1}{R^2} r^3 + \text{ higher order terms in } r$$

   Thus, the Gaussian curvature of S^2_R, $K = \frac{1}{R^2}$, measures the deviation from the Euclidean length formula.



4. Let M be the torus of revolution discussed in Exercise 5.6, p. 124. Show by direct computation that $\iint_M K\,dS = 0$, i.e., the *average* Gaussian curvature is zero. (This, of course, is required by the Gauss-Bonnet theorem.)

5. For the surface M pictured below, what is the value of the integral $\iint_M K\,dS$?



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